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Projective well-orderings and bounded forcing axioms. (English summary)
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In 1977 L. Harrington proved that there is a generic extension of L, the model of all constructible sets, in which Martin's axiom plus the negation of the continuum hypothesis hold and there is a $\mathbf{\Sigma}^1_3$ well-ordering of the reals. In 2000 S. Friedman improved Harrington's theorem by showing that the well-ordering could be Σ^1_3. A different way to strengthen Harrington's theorem lies in requiring stronger forcing axioms. This is the approach followed by the paper under review.

The paper is devoted to proving the following theorem: Assume that $L[\mathcal{E}]$ is a fine structural model with a strong cardinal but without inner models with a Woodin cardinal. Then there is a generic extension of $L[\mathcal{E}]$ where the following holds: (1) the semiproper forcing axiom for posets of cardinality less than $2^{\aleph_0}$ $(\mathrm{SPFA}(2^{\aleph_0}))$ and the bounded semiproper forcing axiom (in fact BSPF^{++}); (2) Woodin's statement ψ_{AC}; (3) all $\mathbf{\Delta}^1_3$ sets of reals are Lebesgue measurable and have the property of Baire; and (4) there exists a Σ^1_6 well-ordering of the reals.

After a brief introduction, the author devotes Section 2 to exposing briefly the main techniques and results of inner model theory which he needs in the proof of the theorem. Section three is devoted entirely to the proof of the theorem. The proof splits up naturally into four parts, one for any statement. The paper finishes with some considerations about possible improvements of the theorem.

Reviewed by *Roger Bosch*

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