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Caicedo, Andrés Eduardo (1-CAIT); **Veličković, Boban** (F-PARIS7-ML)

The bounded proper forcing axiom and well orderings of the reals. (English summary)

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In this paper, the bounded proper forcing axiom BPFA is investigated. BPFA is one of the forcing axioms stronger than $\mathrm{MA}_{\aleph_1}$, and was introduced by M. R. Goldstern and S. Shelah [J. Symbolic Logic **60** (1995), no. 1, 58–73; MR1324501 (96g:03083)]. In [J. Math. Log. **5** (2005), no. 1, 87–97; MR2151584 (2006c:03076)], J. T. Moore proved that BPFA decides the value of the continuum. This was a key question of set theory, whether forcing axioms stronger than $\mathrm{MA}_{\aleph_1}$ decide the value of the continuum. In that paper, he introduced the mapping reflection principle MRP which is deduced from the proper forcing axiom PFA, and showed it by introducing a statement υ_{AC}, which implies that there is a well ordering of the reals of length ω_2 which is Δ_2-definable in the structure $(H_{\aleph_2}, \in)$ with parameter a subset of ω_1. We have to note that although BPFA does not imply MRP, using the idea of MRP, he proved that BPFA implies υ_{AC}.

The following theorems are proved in this paper. (1) If M is an inner model, BPFA holds in both M and the universe, and M calculates ω_2 correctly, then $\mathcal{P}(\omega_1) \subseteq M$. (2) BPFA implies that there is a Δ_1 well ordering of $\mathcal{P}(\omega_1)$ with parameter a subset of ω_1, whose length is ω_2. (1) is a strengthening of the result of the second author proved in [Adv. Math. **94** (1992), no. 2, 256–284; MR1174395 (93k:03045)]. To show (1), ideas in [J. T. Moore, op. cit.] were used and a robust coding of reals by triples of ordinals smaller than ω_2. (2) is a strengthening of the result in [J. T. Moore, op. cit.].

In this paper, there are also applications involving a minimal model of BPFA and the theorem that under PFA, the set of validities of the logic with the Härtig quantifier is not ordinal definable in $L(\mathbb{R})$.

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References

1. J. Bagaria, *Bounded forcing axioms as principles of generic absoluteness*, Arch. Math. Logic **39** (2000), no. 6, 393–401. MR1773776 (2001i:03103)
2. J. Baumgartner, *Iterated forcing*, Iterated forcing. Surveys in set theory, 1–59, London Math. Soc. Lecture Note Ser., 87, Cambridge Univ. Press, Cambridge, 1983. MR0823775 (87c:03099)
3. J. Baumgartner, *Applications of the proper forcing axiom*, Handbook of set-theoretic topology, North-Holland (1984) 913–959. MR0776640 (86g:03084)
4. M. Bekkali, Topics in set theory, Springer (1991). MR1119303 (92m:03070)
5. A. Caicedo, *Projective well-orderings and bounded forcing axioms*, J. Symbolic Logic **70** (2005), no. 2, 557–572. MR2140046 (2006b:03060)
6. A. Caicedo, S. Friedman. *Projective well-orderings of the reals: A survey*, in preparation.
7. M. Foreman, M. Magidor, and S. Shelah, *Martin's maximum, saturated ideals, and nonregular*

ultrafilters. I, Ann. of Math. (2) **127** (1988), no. 1, 1–47. MR0924672 (89f:03043)

8. M. Goldstern and S. Shelah, *The bounded proper forcing axiom*, J. Symbolic Logic **60** (1995), no. 1, 58–73. MR1324501 (96g:03083)
9. H. Herre, M. Krynicky, A. Pinus, and J. Väänänen, *The Härtig quantifier: A survey*, J. Symbolic Logic **56** (1991), no. 4, 1153–1183. MR1136448 (93c:03051)
10. T. Jech, Set theory, the third millenium edition, Springer (2003). MR1940513 (2004g:03071)
11. B. König and Y. Yoshinobu, *Fragments of Martin's Maximum in generic extensions*, MLQ Math. Log. Q. **50** (2004), no. 3, 297–302. MR2050172 (2005b:03117)
12. P. Larson, *The nonstationary ideal in the* $\mathcal{P}_{max}$ *extension*, preprint. cf. MR 2008d:03049
13. D. Martin and R. Solovay, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), no. 2, 143–178. MR0270904 (42 #5787)
14. J. Moore, *Set mapping reflection*, J. Math. Log. **5** (2005), no. 1, 87–98. MR2151584 (2006c:03076)
15. S. Shelah, Proper and improper forcing, Springer (1998). MR1623206 (98m:03002)
16. R. Solovay, *The independence of* DC *from* AD, in Cabal Seminar 76–77 (Proceedings of the UCLA-Caltech logic seminar, 1976–77), A. Kechris, Y. Moschovakis, eds., Springer (1978) 171–183. MR0526918 (80e:03065)
17. R. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. Math. **94** (1971), no. 2, 201–245. MR0294139 (45 #3212)
18. J. Steel, PFA *implies* $\mathrm{AD}^{L(\mathbb{R})}$, preprint. cf. MR 2008b:03069
19. S. Todorčević, *A note on the proper forcing axiom*, in Axiomatic set theory (Boulder, Colorado, 1983), J. Baumgartner, D. Martin, eds., Amer. Math. Soc. (1984), 209–218. MR0763902 (86f:03089)
20. S. Todorčević, *Generic absoluteness and the continuum*, Math. Res. Lett. **9** (2002) 1–7. MR1928866 (2003f:03067)
21. B. Veličković, *Forcing axioms and stationary sets*, Adv. Math. **94** (1992), no. 2, 256–284. MR1174395 (93k:03045)
22. H. Woodin, The axiom of determinacy, forcing axioms, and the nonstationary ideal, Walter de Gruyter (1999). MR1713438 (2001e:03001)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.