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Caicedo, Andrés Eduardo (1-CAIT); **Schindler, Ralf** (D-MUNS-ML)
Projective well-orderings of the reals. (English summary)
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The authors investigate the possible existence of projective well-orderings of the reals in connection with large cardinal hypotheses. From the beginning of set theory, being able to define a well-ordering of the reals has been a focal issue, and through Gödel's work on the constructible universe L it is known to be consistent to have a Σ^1_2 well-ordering of the reals. As a first comment, the authors point out that if the universe is not closed under sharps, then there is a forcing extension with a $\mathbf{\Sigma}^1_2$ well-ordering of the reals.

The authors then show that under the supposition that there is no inner model with ω strong cardinals, there is a forcing extension with a projective well-ordering of the reals, and in fact if n is the number of strong cardinals in the extant core model K, then there is an extension with a $\mathbf{\Delta}^1_{n+3}$ well-ordering of the reals. The proof uses almost disjoint coding and the inner model machinery of the second author in particular.

In counterpoint, the authors describe delimitative results by Woodin. In particular, Woodin established that if there are $n > 0$ strong cardinals, then there is a forcing extension in which there is no $\mathbf{\Sigma}^1_{n+2}$ well-ordering of the reals in any further set generic extension.

These various results establish in broad outline the pivotal nature of having strong cardinals in connection with the complexity along the projective hierarchy for well-orderings of the reals. The authors conclude with several refining open questions.

Reviewed by *A. Kanamori*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.