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Regressive functions on pairs. (English summary)
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Like the Paris-Harrington theorem [J. Paris and L. Harrington, in *Handbook of mathematical logic*, 1133–1142, Studies in Logic and the Foundations of Math., 90, North-Holland, Amsterdam, 1977; see MR0491063 (58 #10343)], the Kanamori-McAloon theorem [A. Kanamori and K. McAloon, Ann. Pure Appl. Logic **33** (1987), no. 1, 23–41; MR0870685 (88i:03095)] is a Ramsey-theoretic statement that, while true, cannot be proved in Peano arithmetic. Say that a coloring $f\colon [X]^k \to \mathbb{N}$, where $X \subset \mathbb{N}$, is regressive if $f(s) < \min(s)$ for all s such that $\min(s) > 0$. Moreover, a subset H of X is min-homogeneous for f if, for every s, t in $[H]^k$, $\min(s) = \min(t)$ implies $f(s) = f(t)$; that is, the coloring induced by f on H only depends on the minimal element in each k-element set. The proposition considered by Kanamori and McAloon then states that for any natural numbers k and n there is an m such that any regressive coloring of $[m]^k$ contains a min-homogeneous set H of size n. This theorem follows by applying a standard compactness argument to the canonical Ramsey theorem of P. Erdős and R. Rado [J. London Math. Soc. **25** (1950), 249–255; MR0037886 (12,322f)].

Let $g(n)$ be the smallest m such that any regressive coloring of $[m]^2$ contains a min-homogeneous set of size n, that is, the function considered above in the specific case of pairs. Using model theory, Kanamori and McAloon also showed that the function $g(n)$ has a growth rate which is at least Ackermannian, eventually dominating every primitive recursive function. An elementary proof of this theorem was given by M. Kojman and S. Shelah [J. Combin. Theory Ser. A **86** (1999), no. 1, 177–181; MR1682971 (2000c:05147)].

In this paper, the author considers a slightly more general problem than Kojman and Shelah's. Let $g(n, l)$ be the smallest m such that, for any regressive function $f\colon [l, m]^2 \to [0, m-2]$, there is a min-homogeneous set for f of size n. The author bounds this function, showing that, for each n, there exists a constant c_n such that

$$A_{n-1}(l-1) \leq g(n, l) \leq A_{n-1}(c_n l),$$

where A_n is the Ackermann function, thus determining the behaviour of the regressive Ramsey function for pairs, as $g(n, l)$ is called, up to the right level of the Ackermann hierarchy. The author concludes with a close study of the values of the function $g(n, l)$ for small values of l.

Reviewed by *David Conlon*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.