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Miller, Arnold W. (1-WI)
A Dedekind finite Borel set. (English summary)
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An infinite set is *Dedekind finite* iff every proper subset is of strictly smaller cardinality. Of course, no such sets exist if one assumes countable choice, but it is well known that their existence is consistent with ZF. Somewhat more surprisingly, one can even show the consistency with ZF of the existence of an infinite Dedekind finite set of reals; an example of such a model is already Cohen's first model. For references, see [P. E. Howard and J. E. Rubin, *Consequences of the axiom of choice*, Math. Surveys Monogr., 59, Amer. Math. Soc., Providence, RI, 1998; MR1637107 (99h:03026)]. (By *reals*, we mean here elements of the Cantor set 2^ω.)

It is well known that not assuming countable choice allows us to obtain other pathologies. For example, in the Feferman-Levy model, the reals are the countable union of countable sets, so every set of reals is Borel [see T. J. Jech, *The axiom of choice*, North-Holland, Amsterdam, 1973; MR0396271 (53 #139)]. On the other hand, in the Feferman-Levy model no infinite set of reals is Dedekind finite, as any countable union of finite subsets of a linear order is countable.

The main result of the paper under review (Theorem 1.4) is that it is consistent with ZF that there is an infinite Dedekind finite Borel (in fact, $F_{\sigma\delta}$) set of reals. First (Lemma 4.2), it is shown in ZF that if $\mathbb{P}$ is a σ-centered poset, then forcing with $\mathbb{P}$ does not destroy any Dedekind finite sets. Starting with a model where there is an infinite Dedekind finite set D of reals, a σ-centered poset is devised (Definition 4.9) using a variant of the well-known almost disjoint sets forcing of R. M. Solovay [see D. A. Martin and R. M. Solovay, Ann. Math. Logic **2** (1970), no. 2, 143–178; MR0270904 (42 #5787)]. The poset $\mathbb{P}$ is defined so that in a suitable symmetric submodel $\mathcal{N}$ of the extension by $\mathbb{P}$ (Definition 4.13), the set D is $F_{\sigma\delta}$ (Lemma 4.18), and therefore $\mathcal{N}$ is the model of the statement under consideration.

It is also shown in the paper that the complexity of the set D cannot be essentially improved. Namely, in Theorem 1.2 it is shown in ZF that $G_{\delta\sigma}$ sets of reals have the perfect set property (so, in particular, they cannot be infinite Dedekind finite). In turn, this cannot be improved much: Theorem 1.3 shows that if the reals are the countable union of countable sets, then there are uncountable $F_{\sigma\delta}$ sets without a perfect subset. Also, it is shown in Remark 3.5 that under the same assumption, there are no universal $F_{\sigma\delta}$ sets (though the existence of universal F_σ sets is provable in ZF).

Remark 3.4 needs to be addressed, since it directly contradicts the statement of Theorem 3.3 (c) of [A. Kanamori and D. Pincus, in *Paul Erdős and his mathematics, II (Budapest, 1999)*, 413–426, Bolyai Soc. Math. Stud., 11, János Bolyai Math. Soc., Budapest, 2002; MR1954736 (2003m:03076)]. Neither paper includes a proof. The author has provided a proof of Remark 3.4 in the short note ["Remark 3.4 a Dedekind finite Borel set", preprint, arXiv:1509.08947], which also includes the correct statement of Theorem 3.3 (c), as verified by Kanamori.

Let me close by saying that the paper is carefully written and easily enjoyable.

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Current version of review. Go to earlier version. *Andrés Eduardo Caicedo*

References

1. Cohen, P.J.: Set theory and the continuum hypothesis. W. A. Benjamin, Inc., New York-Amsterdam (1966) pp. vi+154 MR0232676 (38 #999)
2. Creed, P., Truss, J.K.: On o-amorphous sets. Ann. Pure Appl. Log. **101**(2–3), 185–226 (2000) MR1736062 (2001d:03090)
3. Creed, P., Truss, J.K.: On quasi-amorphous sets. Arch. Math. Log. **40**(8), 581–596 (2001) MR1867682 (2003a:03074)
4. De la Cruz, O.: Finiteness and choice. Fund. Math. **173**(1), 57–76 (2002) MR1899047 (2003d:03080)
5. Downey, R., Slaman Theodore, A.: On co-simple isols and their intersection types. Ann. Pure Appl. Log. **56**(1–3), 221–237 (1992) MR1167695 (93g:03042)
6. Ellentuck, E.: The universal properties of Dedekind finite cardinals. Ann. Math. **82**(2), 225–248 (1965) MR0180494 (31 #4729)
7. Herrlich, H.: Axiom of choice. Lecture Notes in Mathematics, Springer-Verlag, Berlin, (2006). pp. xiv+194. ISBN: 978-3-540-30989-5 MR2243715 (2007f:03069)
8. Howard Paul, E., Yorke Mary, F.: Definitions of finite. Fund. Math. **133**(3), 169–177 (1989) MR1065900 (91h:03068)
9. Jech, T.J.: The Axiom of Choice. Studies in Logic and the Foundations of Mathematics, vol. 75. North-Holland Publishing Co., Amsterdam-London; Amercan Elsevier Publishing Co., Inc., New York (1973), pp. xi+202 MR0396271 (53 #139)
10. Kechris, A.S.: Classical Descriptive set Theory's Graduate Texts in Mathematics, 156. Springer-Verlag, New York (1995,) pp. xviii+402 MR1321597 (96e:03057)
11. Lévy, A.: The independence of various definitions of finiteness. Fund. Math. **46**, 1–13 (1958) MR0098671 (20 #5126)
12. Mendick, G.S., Truss, J.K.: A notion of rank in set theory without choice. Arch. Math. Log. **42**(2), 165–178 (2003) MR1961876 (2005g:03071)
13. Miller Arnold, W.: Long Borel hierarchies. Math. Log. Q. **54**, 301–316 (2008) MR2417803 (2009d:03120)
14. McCarty, C.: Markov's principle, isols and Dedekind finite sets. J. Symb. Log. **53**(4), 1042–1069 (1988) MR0973100 (90c:03056)
15. Maddy, P.: Believing the axioms. I. J. Symb. Log. **53**(2), 481–511 (1988) MR0947855 (89i:03007)
16. Moore, G.H.: Zermelo's Axiom of Choice. Its Origins, Development, and Influence. Studies in the History of Mathematics and Physical Sciences, 8. Springer-Verlag, New York (1982). pp. xiv+410 MR0679315 (85b:01036)
17. Monro, G.P.: Independence results concerning Dedekind-finite sets. J. Austral. Math. Soc. **19**, 35–46 (1975) MR0360268 (50 #12718)
18. Truss, J.K.: Classes of Dedekind finite cardinals. Fund. Math. **84**(3), 187–208 (1974) MR0469760 (57 #9541)
19. Truss, J.K.: The structure of amorphous sets. Ann. Pure Appl. Log. **73**(2), 191–233 (1995) MR1332569 (96i:03047)
20. Walczak-Typke, A.C.: The first-order structure of weakly Dedekind-finite sets. J. Symb. Log. **70**(4), 1161–1170 (2005) MR2194243 (2006j:03072)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.