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**Graphs on Euclidean spaces defined using transcendental distances. (English summary)**
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Given a set $D$ of positive numbers, let $X_n(D)$ denote the graph with vertex set $\mathbb{R}^n$ where two points are joined by an edge precisely when their distance is in $D$. The exact determination of the chromatic number $\mathrm{Chr}(X_n(D))$ is a significant problem, open even for $n=2$, $D=\{1\}$ [see, for example, A. Soifer, *The mathematical coloring book*, Springer, New York, 2009; MR2458293 (2010a:05005)].

Much work has gone also into determining $\mathrm{Chr}(X_n(D))$ for various $D$. The authors give a brief but careful review of known results. Recently, the conjecture that if $D$ is algebraically independent then $\mathrm{Chr}(X_n(D))$ is finite was raised in [B. Bukh, Geom. Funct. Anal. **18** (2008), no. 3, 668–697; MR2438995 (2009m:52037)]. The paper proves a weaker result, namely that for such $D$, $\mathrm{Chr}(X_n(D))$ is countable.

More precisely: Proposition 1 shows that $\mathrm{Chr}(X_1(D))=2$ if $D$ is $\mathbb{Q}$-linearly independent. Theorem 1 shows that if $D$ is algebraically independent $\mathrm{Chr}(X_2(D))$ is countable, and Theorem 2 shows the same for $\mathrm{Chr}(X_n(D))$ for $n\geq 3$. Finally, Theorem 3 shows that for any countable subfield $\mathbb{F}$ of $\mathbb{C}$, and any $D\subseteq\mathbb{C}$ algebraically independent over $\mathbb{Q}$, we have that $\mathrm{Chr}(Y_n)$ is countable, where $Y_n$ is the graph on $\mathbb{C}^n$ where two points $\vec{a},\vec{b}$ are joined by an edge iff there is some polynomial $p(\vec{x},\vec{y})\in\mathbb{F}[\vec{x},\vec{y}]$ with $p(\vec{x},\vec{x})$ identically zero, and $p(\vec{a},\vec{b})\neq 0$ algebraic over some $d\in D\cup\mathbb{F}$.

Besides (basic) algebra, the proofs combine (finite) Ramsey theory and canonical Ramsey theory, with set theoretic and model theoretic arguments. The paper is carefully written. Bukh's general conjecture remains open. *Andrés Eduardo Caicedo*

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