

DOWNWARD TRANSFERENCE OF MICE AND UNIVERSALITY OF LOCAL CORE MODELS

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Abstract. If \mathbf{M} is a proper class inner model of ZFC and $\omega_2^{\mathbf{M}} = \omega_2$, then every sound mouse projecting to ω and not past 0^{\sharp} belongs to \mathbf{M} . In fact, under the assumption that 0^{\sharp} does not belong to \mathbf{M} , $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$ is universal for all countable mice in \mathbf{V} .

Similarly, if \mathbf{M} is a proper class inner model of ZFC, $\delta > \omega_1$ is regular, $(\delta^+)^{\mathbf{M}} = \delta^+$, and in \mathbf{V} there is no proper class inner model with a Woodin cardinal, then $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is universal for all mice in \mathbf{V} of cardinality less than δ .

§1. Introduction.

1.1. Motivation. This paper is the second in a series started with Caicedo [2]. The goal is to study the structure of (not necessarily fine structural) inner models of the set theoretic universe \mathbf{V} , perhaps in the presence of additional axioms, assuming that there is agreement between (some of) the cardinals of \mathbf{V} and of the inner model.

Although the results in this paper do not concern forcing axioms, it was a recently solved problem in the theory of the proper forcing axiom, PFA, that motivated our results.

It was shown in Veličković [26] that if Martin's maximum, MM, holds, and \mathbf{M} is an inner model that computes ω_2 correctly, then $\mathbb{R} \subset \mathbf{M}$. The argument requires the weak reflection principle to hold, a consequence of both MM and PFA⁺. Todorčević [25] showed that the same result holds if, rather than MM, the reflection principle known as Rado's conjecture is assumed.

It is natural to wonder whether PFA suffices as well. A partial result in this direction was shown in Caicedo–Veličković [3], namely that if \mathbf{M} is an inner model that computes ω_2 correctly, and *both* \mathbf{V} and \mathbf{M} satisfy the bounded proper forcing axiom, BPFA, then again $\mathbb{R} \subset \mathbf{M}$.

However, it was recently shown by Friedman [8] (see also Neeman [16]) that the answer to the PFA question is negative: Using a variant of the method of models as side conditions, Friedman defines a forcing notion that, starting with a supercompact κ , produces an extension where PFA holds and κ becomes ω_2 . The generic for the partial order where only the side conditions are considered also collapses κ to size ω_2 , but the resulting extension does not contain all the reals.

Assume that PFA holds in \mathbf{V} , and that \mathbf{M} is an inner model with the same ω_2 . It seems not completely unreasonable to expect that, even though not all reals can be

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guaranteed to be in \mathbf{M} , the reals of $\mathbf{V} \setminus \mathbf{M}$ should be *generic* in some sense. A possible way of formalizing this intuition consists in trying to show that the large cardinal strength coded by reals in \mathbf{V} is also present in \mathbf{M} . When approaching this problem, we realized that there may be some ZFC results that the presence of forcing axioms could be hiding.

To illustrate this, we mention a couple of observations:

- (1) Assume first that \mathbf{M} is an inner model such that $\text{CAR}^{\mathbf{M}} = \text{CAR}$, where CAR denotes the class of cardinals, and that PFA holds. Then, for example, $\text{AD}^{\mathbf{L}(\mathbb{R})}$ holds in \mathbf{M} and in all its set generic extensions, by Steel [22]. The point here is that Steel [22] shows that $\text{AD}^{\mathbf{L}(\mathbb{R})}$ holds in any generic extension of the universe by a forcing of size at most κ , whenever \square_κ fails for κ a strong limit singular cardinal. If PFA holds, the claim follows by recalling that PFA contradicts \square_κ for all uncountable κ , and that \square_κ relativizes up to any outer model where κ and κ^+ are still cardinals. In particular, many mice of \mathbf{V} are in \mathbf{M} . For example, 0^\sharp , the sharp for a model with a strong cardinal, as well as the sharp for a model with a proper class of strong cardinals, are both in \mathbf{M} by absoluteness. (On the other hand, although \mathbf{M} thinks that M_1^\sharp exists, it is not immediately clear whether its version of M_1^\sharp is the M_1^\sharp of \mathbf{V} .)
- (2) If PFA holds and \mathbf{M} is an inner model that satisfies BPFA and computes ω_2 correctly, then $H_{\omega_2} \subset \mathbf{M}$ by Caicedo–Veličković [3], so the setting of Claverie–Schindler [4] applies. Therefore, using notation as in Claverie–Schindler [4], \mathbf{M} is closed under any mouse operator \mathcal{M} that does not go beyond M_1^\sharp .

The moral here is that it seems that we only need to assume a local version of agreement of cardinals rather than a global one to conclude that an inner model \mathbf{M} must absorb any significant large cardinal strength coded by reals in \mathbf{V} .

1.2. Main result. It is interesting to note that if \mathbf{M} computes ω_2 correctly, and 0^\sharp exists, then $0^\sharp \in \mathbf{M}$, see Friedman [7]. We have started a systematic approach to the question of how far this result can be generalized. Using techniques different from those in Friedman [7], we reprove this result, and our method allows us to show that the same conclusion can be obtained with 0^\sharp replaced by stronger mice, see for example Lemma 1.3 and Corollary 1.10.

CONJECTURE 1.1. *Let r be a 1-small sound (iterable) mouse that projects to ω . Assume that \mathbf{M} is an inner model and that $\omega_2^{\mathbf{M}} = \omega_2$. Then $r \in \mathbf{M}$.*

The expected argument to settle Conjecture 1.1 is an induction on the mice hierarchy. The mice as in the conjecture are lined up so that, if it fails, then there is a least counterexample r . Comparing r with $\mathbf{K}^{\mathbf{M}}$, one expects to reach the contradiction that r is in fact an initial segment of $\mathbf{K}^{\mathbf{M}}$. Our result is that this is indeed the case under appropriate anti-large cardinal assumptions, see for instance Corollary 10.

We will consider a proper class inner model \mathbf{M} and the true core model of the universe \mathbf{M} ; we denote this core model by $\mathbf{K}^{\mathbf{M}}$. Of course, we will only consider situations where $\mathbf{K}^{\mathbf{M}}$ exists, so we need to assume that in \mathbf{M} there is no proper class inner model with a Woodin cardinal. We will actually make the assumption that there is no proper class inner model with a Woodin cardinal in the sense of \mathbf{V} . Then proper class extender models of \mathbf{M} fully iterable in the sense of \mathbf{M} are fully iterable in \mathbf{V} , which will play an important role in the proof of our main theorem. Here we

mean iterability with respect to fine structural iteration trees. Also, in this paper we consider the terms “premouse” and “extender model” to be synonymous, although in the literature the term “extender model” often implicitly means that the model in question is a proper class. This explains our use of the term “proper class extender model” above.

The proof of this kind of absoluteness of iterability is standard, and we give it in Section 2. We briefly comment in Section 4 on what the situation is under the weaker hypothesis that there is no proper class inner model with a Woodin cardinal in the sense of \mathbf{M} .

We remark that, although we restrict ourselves to the situation where there are no Woodin cardinals in inner models, we expect that our universality results can be strengthened to reach beyond this point.

We will be using the core model theory as developed in Steel [21]; this approach uses the core model constructed in a universe below a measurable cardinal, which simplifies the exposition. Thus the universe we will be working with will have the property that its ordinals constitute a measurable cardinal in some longer universe. We believe, but did not check carefully, that our arguments for the core model up to a Woodin cardinal can be carried out—in a more or less straightforward way—in the core model theory developed in Jensen–Steel [12] which avoids the use of a measurable cardinal. We have tried to make our exposition as independent of the direct reference to the existence of a measurable cardinal in \mathbf{V} as possible, so that there will be enough heuristic evidence for our belief. Regarding the fine structural aspects, we will use Jensen’s Σ^* -fine structure theory as described in Zeman [28] or Welch [27], and use some notation from Zeman [28]; we believe this notation is self-explanatory. Otherwise our notation will be fully compatible with that in Steel [21].

Following Steel [21], when working with the core model up to a Woodin cardinal, Ω will be reserved to denote the measurable/subtle cardinal needed to develop the basic core model theory in the universe we are working in. By \mathbf{V} , we will mean the rank-initial segment of the universe up to Ω . The notion “class” will then have the corresponding meaning. When working with the core model up to a strong cardinal we let $\Omega = \mathbf{On}$, that is, the class of all ordinals, let \mathbf{V} be the actual universe, and use the core model theory below 0^\sharp , the sharp for a strong cardinal, as described in Zeman [28]. It should be stressed that the proof of our main theorem, Theorem 1.6, will only require to construct $\mathbf{K}^{\mathbf{M}}$, the core model of \mathbf{M} . In this case the universe we will be working in will be \mathbf{M} , so we will work below Ω relative to \mathbf{M} .

Recall the following notation that will be used throughout the paper. We will use Mitchell–Steel indexing of extenders as introduced in Mitchell–Steel [15], see also [23], with the only exception when we work with the core model below 0^\sharp , in which case we use Jensen’s indexing as described in Chapter 8 in Zeman [28]. We will make it explicit in the text when we use Jensen’s indexing. If N is any extender model and $\alpha \leq \text{ht}(N)$ is an ordinal, then $N \parallel \alpha$ is the extender model $\langle J_\alpha^E, E_{\omega\alpha} \rangle$ where E is the extender sequence of N . If α does not index an extender in N , for instance when α is a cardinal in N , then $E_{\omega\alpha} = \emptyset$ and in this case we identify $N \parallel \alpha$ with J_α^E . Recall also that if \mathcal{T} is a normal iteration tree then $\delta(\mathcal{T})$ is the supremum of all natural lengths of extenders used in the tree; see the beginning of Section 2 for more details. If \mathcal{T} is an iteration tree of successor length then the

main branch of the tree is the branch connecting the first model of the tree with its last model.

The following lemma is a particular case of a well-known result of Shelah, see for example Abraham [1]. The point of this short detour is to explain why we introduce the set S_δ below and concentrate on it rather than directly assuming agreement of cardinals, as discussed above. We use the Foreman–Magidor definition of $\mathcal{P}_\kappa(\lambda)$ from Foreman–Magidor [6], so

$$\mathcal{P}_\kappa(\lambda) = \{x \subseteq \lambda \mid \text{otp}(x) < \kappa \ \& \ x \cap \kappa \text{ is transitive.}\}. \tag{1}$$

The notion of stationarity is that in Woodin’s sense, that is, a set $S \subseteq \mathcal{P}(Z)$ is stationary iff for every partial map $f : {}^{<\omega}Z \rightarrow Z$ there is some $x \in X$ which is closed under f .

LEMMA 1.2. *Let \mathbf{M} be a proper class inner model of ZFC such that $\omega_2^{\mathbf{M}} = \omega_2$. Then $\mathcal{P}_{\omega_1}(\omega_2) \cap \mathbf{M}$ is stationary.*

PROOF. Let $f : {}^{<\omega}\omega_2 \rightarrow \omega_2$, and pick $\gamma < \omega_2$ of size ω_1 which is closed under f . Since \mathbf{M} computes ω_2 correctly, it sees a bijection $f : \omega_1 \rightarrow \gamma$. It is then easy to find an $\alpha < \omega_1$ such that $f[\alpha]$ is closed under f and $f[\alpha] \cap \omega_1 = \alpha$. \dashv

To illustrate how the assumption of stationarity of $\mathcal{P}_{\omega_1}(\omega_2) \cap \mathbf{M}$ can be used, consider the following:

LEMMA 1.3. *Let \mathbf{M} be a proper class inner model of ZFC such that $\mathcal{P}_{\omega_1}(\omega_2) \cap \mathbf{M}$ is stationary. Suppose that 0^\sharp exists. Then $0^\sharp \in \mathbf{M}$.*

PROOF. Assume P is the (unique, sound) mouse representing 0^\sharp , so

$$P = (J_\beta, \in, F),$$

where F is an extender representing an amenable measure on an ordinal μ and β is the Mitchell–Steel index of F , that is, $\beta = \mu^{++L}$. (See the beginning of Section 2 for more details on Mitchell–Steel indexing.) Furthermore, P has first projectum $\rho_P^1 = \omega$. Let θ be regular and large. By assumption, we can find some countable $X \preceq H_\theta$ such that:

- (1) $P \in X$,
- (2) $X \cap \omega_2 \in \mathbf{M}$, and
- (3) $\kappa = X \cap \omega_1 \in \omega_1$.

Let H be the transitive collapse of X , and let

$$\sigma : H \rightarrow H_\theta$$

be the inverse of the collapsing map. Then the critical point of σ is κ , $\sigma(\kappa) = \omega_1$, and $P \in H$.

Let P' be the κ -th iterate of P , formed by applying the ultrapower construction using F and its images under the corresponding embeddings. Then $P' \in H$ and also

$$\mathcal{P}(\kappa) \cap \mathbf{L} \in H \cap \mathbf{L}.$$

Note that $P' = (J_{\beta'}, F')$, where β' is the double cardinal successor of $\text{cr}(F')$ in \mathbf{L} , and that $\sigma \upharpoonright J_{\beta'} \in \mathbf{M}$.

By the argument from the proof of the comparison lemma,

$$\sigma \upharpoonright P' = \pi_{\kappa, \omega_1},$$

where the map on the right side is the iteration map. Hence F' is the \mathbf{L} -extender on κ derived from $\sigma \upharpoonright J_{\beta'}$. It follows that $F' \in \mathbf{M}$. But then $P \in \mathbf{M}$. \dashv

Of course, Lemma 1.2 generalizes straightforwardly to larger cardinals, which suggests that rather than looking at direct generalizations of Lemma 1.3 to stronger countable mice, one should instead focus on instances of *local universality*. We will work only with 1-small premice/extender models, and such premice/extender models have at most one iteration strategy, where branches for trees of limit length are given by Q -structures which are levels of \mathbf{L} (see Steel [23] for the notion of Q -structure). Such a Q -structure for \mathcal{T} may not exist, but this only happens in the case where no branch model has an ordinal indexing an extender above $\delta(\mathcal{T})$, which means that any such branch model is a last model of the comparison. We call the strategy described above the uniqueness strategy. When we say a premouse/extender model M is iterable we mean M is iterable via its uniqueness strategy. Recall that a coiteration of 1-small premice M, N is a pair of normal iteration trees $(\mathcal{T}, \mathcal{U})$ according to the uniqueness strategies where \mathcal{T} is on M , \mathcal{U} is on N and the successor steps $\alpha + 1$ in the trees are determined by the extenders which constitute the least disagreements between the α -th models of \mathcal{T} and \mathcal{U} . We say that a coiteration $(\mathcal{T}, \mathcal{U})$ is terminal if and only if the trees \mathcal{T}, \mathcal{U} have last models, and the last model of \mathcal{T} is an initial segment of the last model of \mathcal{U} or vice versa.

DEFINITION 1.4. Let δ be a cardinal, or $\delta = \mathbf{On}$. Let W be an iterable 1-small extender model.

We say that W is *universal* for extender models of size less than δ if and only if W does not lose the coiteration with any iterable 1-small extender model N of size less than δ , that is: Let $(\mathcal{T}, \mathcal{U})$ be the pair of iteration trees according to the uniqueness strategies corresponding to the terminal coiteration of W with N , where \mathcal{T} is on W and \mathcal{U} is on N . Then \mathcal{T}, \mathcal{U} are of length less than δ , there is no truncation on the main branch of \mathcal{U} , and the last model of \mathcal{U} is an initial segment of the last model of \mathcal{T} .

We also fix the following notation. Given a regular cardinal δ , we let

$$S_\delta = \{x \in \mathcal{P}_\delta(\delta^+) \cap \mathbf{M} \mid \text{cf}^{\mathbf{M}}(x \cap \delta) > \omega\}. \tag{2}$$

Here the cardinal successor δ^+ is computed in \mathbf{V} . Note that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ and S_δ are elements of \mathbf{M} . We then have the following variant of Lemma 1.2, whose the proof is an almost literal repetition of the previous one.

LEMMA 1.5. Let $\delta > \omega_1$ be a regular cardinal and \mathbf{M} be a proper class inner model such that $\delta^{+\mathbf{M}} = \delta^+$. Then S_δ is stationary.

Using the terminology from the definition above we can state our main theorem:

THEOREM 1.6. Assume that \mathbf{M} is a proper class inner model, and that δ is a regular cardinal in the sense of \mathbf{V} .

- (a) Granting that, in \mathbf{V} , there is no proper class inner model with a Woodin cardinal, $\delta > \omega_1$, and S_δ is stationary, then the initial segment $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is universal for all iterable premice in \mathbf{V} of cardinality less than δ .
- (b) Granting that, in \mathbf{M} , 0^\sharp does not exist, and $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary, then: If $\delta > \omega_1$, then the initial segment $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is universal for all iterable premice in \mathbf{V} of cardinality less than δ . Moreover, if $\delta = \omega_1$, then $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$ is universal for all countable iterable premice in \mathbf{V} .

It was pointed out to us by the referee that our arguments can be run under somewhat more relaxed assumptions with only minor amendments. The referee suggested an improvement to Theorem 1.6, which we include in Section 3.5.

The proof of Theorem 1.6 is the heart of this paper and occupies Section 3. Given Lemma 1.5, one gets the obvious variant of Theorem 1.6 where the assumption on stationarity of S_δ is replaced with $\delta^{+\mathbf{M}} = \delta^+$. Part (b) of the theorem gives the same kind of conclusion as (a), but includes the important case $\delta = \omega_1$ which is missing in (a), and uses a hypothesis on $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ which is weaker than that in (a). The cost here is a more restrictive anti-large cardinal hypothesis. We believe part (b) can be merged with part (a) in the sense that the anti-large cardinal assumption in (b) can be replaced with that in (a) (and part (a) can then be deleted), but at the moment we do not see how to run the arguments in the proof of (b) under the anti-large cardinal assumption in (a). On the other hand, the conclusion in (b) that $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$ is universal for countable mice in \mathbf{V} cannot be improved by replacing ω_2 by ω_1 . In fact, such conclusion can be false even if $\mathbf{M} = \mathbf{V}$, by a result of Jensen [10]. For example, it is consistent (under mild large cardinal assumptions) that there is a countable mouse m that iterates past $\mathbf{K} \parallel \omega_1$. In fact, starting with $\mathbf{K} = \mathbf{L}[E]$ such that the universe of $\mathbf{L}[E]$ is the same as that of $\mathbf{L}[U]$ for some normal measure U on some κ , Jensen [10] describes a forcing extension of \mathbf{K} that turns κ into ω_1 , and adds a countable mouse m such that (J_β^E, E_β) is an iterate of m ; here β is the index of the extender coding the measure U . In Section 4 we describe a scenario for a proof of the strengthening of Theorem 1.6(a) where the assumption on stationarity of S_δ is replaced with the assumption that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary, and also discuss a variant of the theorem where the assumption on nonexistence of inner models with Woodin cardinals is weakened to nonexistence of such inner models from the point of view of \mathbf{M} .

1.3. Applications.

COROLLARY 1.7. *Assume that \mathbf{M} is a proper class inner model.*

- (a) *Granting that, in \mathbf{V} , there is no proper class inner model with a Woodin cardinal, and S_δ is stationary for proper class many cardinals δ that are regular in \mathbf{V} , then the model $\mathbf{K}^{\mathbf{M}}$ is universal for all set sized iterable 1-small premice in \mathbf{V} .*
- (b) *Under the assumptions that, in \mathbf{V} , there is no proper class inner model with a strong cardinal, and $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary for proper class many cardinals δ that are regular in \mathbf{V} , we conclude that $\mathbf{K}^{\mathbf{M}}$ is universal with respect to all extender models in \mathbf{V} so, in particular, the models $\mathbf{K}^{\mathbf{M}}$ and $\mathbf{K} = \mathbf{K}^{\mathbf{V}}$ are Dodd–Jensen equivalent.*

PROOF. The conclusions in (a) are obvious consequences of Theorem 1.6. The conclusion on universality of $\mathbf{K}^{\mathbf{M}}$ in (b) follows from the well-known fact that in the absence of proper class inner models with strong cardinals, universality with respect to set-sized extender models implies universality with respect to proper class extender models (see, for example, Zeman [28, Lemma 6.6.6]). Since $\mathbf{K}^{\mathbf{M}}$ and \mathbf{K} are both universal in the sense of \mathbf{V} , they are Dodd–Jensen equivalent. \dashv

If there is no proper class inner model with a strong cardinal then the above corollary is a generalization of the well-known fact that \mathbf{K} is generically absolute; see Zeman [28]. In the case of generic absoluteness we let \mathbf{V} play the role of $\mathbf{M}[G]$ where G is a generic filter for some poset in \mathbf{M} , and conclude that $\mathbf{K}^{\mathbf{M}} = \mathbf{K}$. If \mathbf{V}

is not a generic extension of \mathbf{M} we of course cannot expect that the core model will remain unchanged, but by the above corollary we still can expect that the core models of \mathbf{M} and \mathbf{V} are close.

A well-known example which illustrates the situation where \mathbf{V} is not a generic extension of \mathbf{M} is the following: The model \mathbf{M} is obtained as an ultrapower of \mathbf{V} via some extender in \mathbf{V} . Then $\mathbf{K}^{\mathbf{M}}$ is distinct from \mathbf{K} , yet, if for instance 0^\sharp does not exist, $\mathbf{K}^{\mathbf{M}}$ is a normal iterate of \mathbf{K} ; see again Zeman [28]. Notice that the hypothesis of the above corollary is satisfied here, as δ can be taken to be the cardinal successor of any singular strong limit cardinal of sufficiently high cofinality, assuming there is no proper class inner model with a Woodin cardinal. In this case $\delta^{+\mathbf{M}} = \delta^+$ which guarantees that S_δ is stationary in \mathbf{V} by Lemma 1.5. Thus, the situation just described is an instance of Corollary 1.7.

It is well known that if there is an inner model with a strong cardinal then the conclusion in Corollary 1.7(b) is no longer true, that is, it may happen that $\mathbf{K}^{\mathbf{M}}$ is strictly below \mathbf{K} in the Dodd–Jensen pre-well-ordering, yet \mathbf{M} computes cardinal successors correctly for proper class many regular cardinals in \mathbf{V} . This example is essentially described in Steel [21, Section 3]: It suffices to let \mathbf{V} be the minimal iterable proper class extender model with one strong cardinal, say κ , and construct a linear iteration of length \mathbf{On} by iteratively applying sufficiently large extenders on the images of the extender sequence of \mathbf{V} . Then let \mathbf{M} be the initial segment of the resulting direct limit of ordinal length. We described this example here because it also shows that we cannot expect to prove universality of $\mathbf{K}^{\mathbf{M}}$ with respect to proper class extender models in \mathbf{V} once \mathbf{V} sees proper class inner models with strong cardinals. In the example just described, \mathbf{V} sees a proper class inner model with a strong cardinal, namely itself, but \mathbf{M} does not see any such inner model.

Recall that, by Lemma 1.5, the conclusions in Theorem 1.6(b) hold if the assumption of stationarity of S_δ is replaced with the assumption that \mathbf{M} computes the cardinal successor of δ correctly. For $\delta = \omega_1$, this assumption follows from the requirement that $\omega_2^{\mathbf{M}} = \omega_2^{\mathbf{V}}$. In contrast to this, notice that if 0^\sharp exists and g is $\text{Coll}(\omega, < \omega_1^{\mathbf{V}})$ -generic over \mathbf{V} , then $\mathbf{M} = \mathbf{L}[g]$ is an inner model of $\mathbf{V}[g]$ which computes ω_1 correctly, yet $0^\sharp \notin \mathbf{M}$. There is also a finer folklore result that traces back at least to Hjorth’s thesis (see the Claim on page 430 of Hjorth [9]) that not much can be concluded about the mice of an inner model \mathbf{M} if merely $\omega_1^{\mathbf{M}} = \omega_1$ is assumed. In particular, the following holds:

FACT 1.8. *If 0^\sharp exists, then there exists an inner model \mathbf{M} which is a set forcing extension of \mathbf{L} and computes ω_1 correctly. Thus $0^\sharp \notin \mathbf{M}$.*

QUESTION 1.9. *Is there a class forcing extension of \mathbf{L} such that no inner model with the same ω_1 is a set forcing extension of \mathbf{L} ?*

On the other hand, as hinted by Lemma 1.3, the assumption $\omega_2^{\mathbf{M}} = \omega_2$ is significantly different. The following corollary is another piece of evidence supporting this. The conclusions of the corollary hold under the assumption that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary for some regular $\delta \geq \omega_1$, but we stated it for $\delta = \omega_1$ as we find this instance finest.

COROLLARY 1.10. *Assume that \mathbf{M} is a proper class inner model, and that $\mathcal{P}_{\omega_1}(\omega_2) \cap \mathbf{M}$ is stationary. Let r be a sound mouse in \mathbf{V} projecting to ω such that either r is below 0^\sharp or $r = 0^\sharp$. Then $r \in \mathbf{M}$.*

PROOF. This follows from Theorem 1.6: Assume that the conclusion fails, so $\mathbf{K}^{\mathbf{M}}$ exists and does not reach 0^{\sharp} . Then, by Theorem 1.6(b), $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$ is universal for countable mice in \mathbf{V} , and it follows from a standard comparison argument that such mice must be an initial segment of $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$. \dashv

We reiterate that we made no anti-large cardinal assumptions on \mathbf{V} in Corollary 1.10. Naturally, our results relativize so, for example, for any real $x \in \mathbf{M}$, Corollary 1.10 holds for countable x -mice below x^{\sharp} . Finally we stress that, as already mentioned in the above corollary, the proof of Theorem 1.6(b) actually shows the following:

If 0^{\sharp} exists in \mathbf{V} , then it is actually in \mathbf{M} .

COROLLARY 1.11. *Assume that \mathbf{M} is a proper class inner model, and that δ is a regular cardinal in the sense of \mathbf{V} . Assume further that one of the following holds:*

- (a) *In the sense of \mathbf{V} , there is no inner model with a Woodin cardinal, a^{\dagger} exists for every real a in \mathbf{V} , $\delta > \omega_1$, and S_{δ} is stationary.*
- (b) *In \mathbf{V} , 0^{\sharp} does not exist, $\mathbb{R}^{\mathbf{V}}$ is closed under sharps, $\delta \geq \omega_1$, and $\mathcal{P}_{\delta}(\delta^+) \cap \mathbf{M}$ is stationary.*

Then \mathbf{M} is Σ_3^1 -correct.

PROOF. Let $A \subseteq \mathbb{R}$ be a nonempty Π_2^1 -set. By Schindler [19] under assumption (a) and by Steel–Welch [24] or Schindler [20] under assumption (b), there is a premouse M such that $A \cap M$ is nonempty; pick an r in this intersection. Without loss of generality we may assume M is sound and projects to ω , hence is countable. By Theorem 1.6, the model $\mathbf{K}^{\mathbf{M}} \parallel \max(\delta, \omega_2)$ iterates past M . We conclude that $r \in \mathbf{K}^{\mathbf{M}} \subseteq \mathbf{M}$, so $A \cap \mathbf{M}$ is nonempty. \dashv

Instead of quoting Schindler [19] in the above proof we could alternatively quote Theorem 7.9 in Steel [21]; which, applied inside the model r^{\dagger} for $r \in A$, yields a mouse M as in the proof. By Theorem 1.6, in Corollary 1.11(b) we could equivalently assume that $0^{\sharp} \notin \mathbf{M}$, as such a sharp, if it exists in \mathbf{V} , must already be in \mathbf{M} (as pointed out above). Also, similarly as with Theorem 1.6, appealing to Lemma 1.5, there is an obvious variant of the above corollary where the assumption of stationarity of S_{δ} , resp. $\mathcal{P}_{\delta}(\delta^+) \cap \mathbf{M}$ is replaced with the requirement that $\delta^{+\mathbf{M}} = \delta^+$.

§2. Iterability and background core model theory. In this section we state facts from core model theory relevant to the proof of our main theorem, Theorem 1.6, and establish absoluteness of iterability for proper class extender models between two proper class inner models. The proof is standard, and we give it to make this paper self-contained. We also draw two useful corollaries, not needed in our main argument.

Let us make some remarks concerning notation. As already mentioned in the introduction, we will be using Mitchell–Steel indexing of extenders in order to stay close enough to our primary reference Steel [21]. If M is an extender model, E^M will denote its extender sequence, and E_{β}^M will denote the extender on the sequence E^M indexed by the ordinal β ; here β is the cardinal successor of the natural length of E_{β}^M as calculated in $\text{Ult}(M, E_{\beta}^M)$. Recall that if \mathcal{T} is an iteration tree then $E_{\alpha}^{\mathcal{T}}$ is the extender coming from the α -th model $M_{\alpha}^{\mathcal{T}}$ used to form an ultrapower in \mathcal{T} . If b is a branch through \mathcal{T} , we write $\pi_b^{\mathcal{T}}$ to denote the iteration map along b if the map exists.

If $i \in b$, we write $\pi_{i,b}^T$ to denote the iteration map along the portion of b above i (including i) if the map exists. Thus $\pi_b^T = \pi_{0,b}^T$ and $\pi_{i,b}^T$ exists iff no truncations occur on b above i . The domain of $\pi_{i,b}^T$ is either M_i^T or the result of the corresponding truncation of \mathcal{M}_i^T . Recall also that if \mathcal{T} is a normal tree then $\delta(\mathcal{T})$ is the supremum of all ordinals $\rho(E_\alpha^T)$; here $\rho(E_\alpha^T)$ is the strict supremum of the generators of E_α^T . Next, $E(\mathcal{T})$ is the union of all extender sequences $E^{M_\alpha^T} \upharpoonright \rho(E_\alpha^T)$ where $\alpha < \text{lh}(\mathcal{T})$, and $M(\mathcal{T})$ is the premouse of height $\delta(\mathcal{T})$ whose extender sequence is $E(\mathcal{T})$. If \mathcal{T} is of limit length then $\delta(\mathcal{T})$ is the supremum of all lengths $\text{lh}(E_\alpha^T)$ of extenders used in \mathcal{T} , or equivalently, all iteration indices in \mathcal{T} .

Recall also that if \mathcal{T} is an iteration tree of limit length and b, c are two distinct cofinal well-founded branches through \mathcal{T} , then $\delta(\mathcal{T})$ is a Woodin cardinal in $M_b^T \cap M_c^T$ where M_b^T, M_c^T are the models at the end of the corresponding branches; see for instance Mitchell–Steel [15]. A similar fact is true even in the case where the branch models $\mathcal{M}_b^T, \mathcal{M}_c^T$ are not well-founded, but $\delta(\mathcal{T}) \in \text{wfp}(M_b^T) \cap \text{wfp}(M_c^T)$ where $\text{wfp}(\mathcal{M}_b^T)$ denotes the transitivity maximal well-founded initial segment of \mathcal{M}_b^T , and similarly for \mathcal{M}_c^T . In this case, $\delta(\mathcal{T})$ has the Woodinness property with respect to all subsets of $\delta(\mathcal{T})$ which are in $\mathcal{M}_b^T \cap \mathcal{M}_c^T$. Recall from the introduction that, granting there is no proper class inner model with a Woodin cardinal, any iterable proper class extender model has precisely one iteration strategy called the *uniqueness* strategy; this is the strategy that picks the unique cofinal well-founded branch at each limit step of the iteration.

The following lemma is a slight variation of Steel [21, Lemma 5.12], and may be viewed as a generalization thereof, as we allow \mathbf{V} be any outer extension of \mathbf{M} , and not merely a generic extension. The proof thus requires an additional appeal to Shoenfield absoluteness, which somewhat changes not only the structure of the proof but the statement of the result itself. We give the proof below; one notable difference is that the proof of Steel [21, Lemma 5.12] uses, in an important way, the assumption that every set in the ground model has a sharp. The existence of sharps is, of course, an immediate consequence of the initial setting where Ω is a measurable cardinal and our universe of interest is the rank-initial segment of height Ω . However, this assumption may be false if we want to run the proof of Theorem 1.6 in ZFC alone. There is one more difference between the two proofs which makes our lemma in a sense “less general” than Steel [21, Lemma 5.12], as we are making the assumption that there is no inner model with a Woodin cardinal in the sense of \mathbf{V} . The proof of Steel [21, Lemma 5.12] only requires that no Woodin cardinals exist in the extender model we want to iterate. We do not know if the assumption on the nonexistence of an inner model with a Woodin cardinal in the sense of \mathbf{V} can be weakened by replacing \mathbf{V} with the model we intend to iterate or with our inner model \mathbf{M} , even in the case where we allow our universe to be closed under sharps.

LEMMA 2.1. *Assume that \mathbf{M} is a proper class inner model, and that in \mathbf{V} there is no proper class inner model with a Woodin cardinal. In \mathbf{M} , let R be a proper class extender model. The following are equivalent:*

- (a) *In \mathbf{M} , the model R is normally iterable via the uniqueness strategy.*
- (b) *In \mathbf{V} , the model R is normally iterable via the uniqueness strategy.*

Here “iterable” means iterable with respect to trees of set-sized length.

PROOF. We give a proof of the implication (a) \Rightarrow (b). The converse is proved in a similar way, and the argument is much simpler. Toward a contradiction, assume that in \mathbf{V} there is a normal iteration tree \mathcal{T} on R witnessing that (a) \Rightarrow (b) is false. We focus on the essential case where the length of \mathcal{T} is a limit ordinal. Thus, in \mathbf{V} the tree \mathcal{T} does not have a unique cofinal well-founded branch. Using a reflection argument, we can find a regular cardinal $\alpha > \delta(\mathcal{T})^+$ such that \mathcal{T} , when viewed as a tree on $R' = R \parallel \alpha$, does not have a unique cofinal well-founded branch in \mathbf{V} . Notice that the transitive closure of \mathcal{T} is then of size α . Let $\theta > \alpha$ be a limit cardinal with large cofinality, and let T be the lightface theory of $H_\theta^{\mathbf{V}}$ in the language of ZFC. Let $\varphi(x, y, z)$ be the conjunction of the following statements:

- x is a transitive model of T .
- $z, y \in x$, and z is a regular cardinal in x .
- There is a normal iteration tree $\mathcal{U} \in x$ on the premouse y of limit length such that $\text{card}(\text{trcl}(\mathcal{U}))^x = z$, $\delta(\mathcal{U})^{+x} < z$, and in x , the tree \mathcal{U} does not have a unique cofinal well-founded branch.
- There is no $a \in x$ such that a is a bounded subset of z and $J_z[a]$ is a model with a Woodin cardinal.

Here $\text{trcl}(v)$ denotes the transitive closure of v .

Let $\beta \geq \text{card}(H_\theta)$, and let G be a filter generic for $\text{col}(\omega, \beta)$ over \mathbf{V} . Immediately from the properties of \mathcal{T} and from our assumption that no inner model with a Woodin cardinal exists in the sense of \mathbf{V} we conclude that $\varphi(H_\theta^{\mathbf{V}}, R', \alpha)$ holds in $\mathbf{V}[G]$. As $H_\theta^{\mathbf{V}}$ is countable in $\mathbf{V}[G]$ and R', α are countable in $\mathbf{M}[G]$, it follows by Shoenfield absoluteness that in $\mathbf{M}[G]$ there is some H such that $\mathbf{M}[G] \models \varphi(H, R', \alpha)$.

Pick a $\text{col}(\omega, \beta)$ -name $\dot{H} \in \mathbf{M}$ and some regular θ' such that $H = \dot{H}_G$ and $\beta, \dot{H}, R' \in H_{\theta'}^{\mathbf{M}}$. Finally, pick a condition $p \in \text{col}(\omega, \beta)$ such that p forces $\varphi(\dot{H}, \dot{R}', \dot{\alpha})$ over $H_{\theta'}^{\mathbf{M}}$. From now on, work in \mathbf{M} . Let X be a countable elementary substructure of $H_{\theta'}^{\mathbf{M}}$ such that $\dot{H}, R', \beta, p \in X$, and let

$$\sigma : \tilde{H} \rightarrow H_{\theta'}^{\mathbf{M}}$$

be the inverse of the Mostowski collapsing isomorphism that arises from collapsing X . Let $\tilde{H}, \tilde{R}, \tilde{\alpha}, \tilde{\beta}, \tilde{p}$ be the preimages of $\dot{H}, R', \alpha, \beta, p$ under σ . Pick a filter g extending \tilde{p} that is generic for $\text{col}(\omega, \tilde{\beta})$ over \tilde{H} , and let $\tilde{H} = \tilde{H}_g$. Then $\tilde{\alpha}$ is a regular cardinal in \tilde{H} , and there is a normal iteration tree $\tilde{\mathcal{U}} \in \tilde{H}$ on \tilde{R} of limit length such that $\text{card}(\text{trcl}(\tilde{\mathcal{U}}))^{\tilde{H}} = \tilde{\alpha}$, $\delta(\tilde{\mathcal{U}})^{+\tilde{H}} < \tilde{\alpha}$ and, in \tilde{H} , the tree $\tilde{\mathcal{U}}$ does not have a unique cofinal well-founded branch.

We next observe that for every limit ordinal $\lambda^* < \tilde{\lambda}$, the tree $\tilde{\mathcal{U}}$ has at most one cofinal well-founded branch in \mathbf{M} : If $b \neq c$ were two distinct cofinal well-founded branches through $\tilde{\mathcal{U}} \upharpoonright \lambda^*$, then $\delta(\tilde{\mathcal{U}} \upharpoonright \lambda^*)$ would be Woodin in $M_b^{\tilde{\mathcal{U}}} \cap M_c^{\tilde{\mathcal{U}}}$. If $\nu > \delta(\tilde{\mathcal{U}} \upharpoonright \lambda^*)$ is least among all ordinals indexing extenders in $M_b^{\tilde{\mathcal{U}}}$ or $M_c^{\tilde{\mathcal{U}}}$ (note such a ν always exists as $\lambda^* < \tilde{\lambda}$), assume without loss of generality that ν indexes an extender in $M_b^{\tilde{\mathcal{U}}}$. Then $\delta(\tilde{\mathcal{U}} \upharpoonright \lambda^*)$ is Woodin in $M_b^{\tilde{\mathcal{U}}} \parallel \nu$ and, using the elementarity of the iteration maps, we conclude that there are $\delta' < \nu'$ such that ν' indexes an extender in \tilde{R} , and λ' is Woodin in $\tilde{R} \parallel \nu'$. Since $\tilde{R} \in \tilde{H}$, we may apply σ to \tilde{R} and conclude that $\sigma(\nu')$ indexes an extender in R' and $\sigma(\delta')$ is Woodin in $R' \parallel \sigma(\nu') = R \parallel \sigma(\nu')$. Since we are assuming R is iterable in the sense of \mathbf{M} ,

iterating the extender on the R -sequence with index ν through the ordinals will produce a proper class inner model with a Woodin cardinal, a contradiction.

The same argument for $\lambda^* = \bar{\lambda}$ shows that in \mathbf{M} , if $b \neq c$ are two cofinal well-founded branches through \bar{U} then no $\nu > \delta(\bar{U})$ indexes an extender in $M_b^{\bar{U}}$ or $M_c^{\bar{U}}$. Now at most one of the branches b, c involves a truncation; otherwise the standard argument can be used to produce two compatible extenders applied in \bar{U} , and thereby obtain a contradiction. If one of the branches involves a truncation, let it be without loss of generality c ; then $M_b^{\bar{U}} \trianglelefteq M_c^{\bar{U}}$. If neither of the branches involves a truncation, let b be such that $M_b^{\bar{U}} \trianglelefteq M_c^{\bar{U}}$. In either case, $\bar{\lambda}$ is Woodin in $M_b^{\bar{U}}$ and there is an iteration map $\pi_b : \bar{R} \rightarrow M_b^{\bar{U}}$. By the elementarity of π_b , there is a Woodin cardinal in \bar{R} . And by the elementarity of σ , there is a Woodin cardinal in R' . Since $R' = R \parallel \alpha$ and α is a cardinal, R is a proper class model with a Woodin cardinal, a contradiction. In particular, this shows that \bar{U} has no cofinal well-founded branch in \bar{H} . (Because our initial hypothesis stipulates that such a branch cannot be unique.)

Since $\sigma \upharpoonright \bar{R} : \bar{R} \rightarrow R'$ is elementary and \bar{U} picks unique cofinal well-founded branches, we can copy \bar{U} onto an R' -based iteration tree U' via $\sigma \upharpoonright \bar{R}$. Let b be a cofinal well-founded branch through U' ; such a branch exists by our assumption on the iterability of R in \mathbf{M} . By the properties of the copying construction, b is also a cofinal well-founded branch through \bar{U} . Now assume $\gamma \geq \bar{\alpha}$ is an ordinal in \bar{H} , and $h \in \mathbf{M}$ is a filter on $\text{col}(\omega, \gamma)$ that is generic over \bar{H} . By our initial setting, $\text{card}(\text{trcl}(U))^{H} = \bar{\alpha} \leq \gamma$; hence both γ and the transitive closure of \bar{U} are countable in $\bar{H}[h]$. Since the statement “There is a cofinal branch x through \bar{U} such that the ordinals of $M_x^{\bar{U}}$ have an initial segment isomorphic to γ ” is Σ_1^1 in the codes, if \bar{U} has such a branch, then some such branch c exists in $\bar{H}[h]$. If $M_c^{\bar{U}}$ is well-founded, then by the previous paragraph, c is unique, and by the homogeneity of $\text{col}(\omega, \gamma)$, such a branch must be in \bar{H} , which contradicts the last conclusion of the previous paragraph. Hence $M_c^{\bar{U}}$ must be ill-founded. If $\mathbf{On} \cap M_b^{\bar{U}} < \mathbf{On} \cap \bar{H}$ put $\gamma = \mathbf{On} \cap M_b^{\bar{U}}$, and apply the above argument to the Σ_1^1 -statement “There is a cofinal branch x through \bar{U} such that the ordinals of $M_x^{\bar{U}}$ are isomorphic to γ ”. We obtain a cofinal well-founded branch through \bar{U} in $\bar{H}[h]$, a contradiction. (Here we are using the fact that there are unboundedly many cardinals in \bar{H} .) It follows that $M_b^{\bar{U}}$ has height at least that of \bar{H} .

Let $\tau = \delta(\bar{U})^{+L[E(\bar{U})]}$. Set $\gamma = \alpha$, and let $h \in \mathbf{M}$ be generic for $\text{col}(\omega, \gamma)$ over \bar{H} . The arguments from the previous paragraph imply that there is a branch $c \in \bar{H}[h]$ that has an initial segment isomorphic to γ . As follows from the previous paragraph, such a branch is ill-founded. Let γ' be a cardinal in \bar{H} larger than the ordinal of the well-founded part of $M_c^{\bar{U}}$, and let h' be a filter generic for $\text{col}(\omega, \gamma')$ over \bar{H} . As $\mathbf{On} \cap M_b^{\bar{U}} > \gamma'$, as before we conclude that there is a branch $c' \in \bar{H}[h']$ with an initial segment isomorphic to γ' . (Here we again make use of the fact that cardinals of \bar{H} are unbounded in \bar{H} .)

Since the well-founded part of $M_c^{\bar{U}}$ is strictly longer than the well-founded part of $M_{c'}^{\bar{U}}$, we conclude that $c \neq c'$. And since both γ, γ' are larger than $\alpha > \tau$, the ordinal $\delta(\bar{U})$ is a Woodin cardinal in $J_\alpha[E(\bar{U})]$. Since $\bar{U} \in \bar{H}$, so is $J_\alpha[E(\bar{U})]$, but this contradicts the last clause in φ which holds in \bar{H} by construction. \dashv

As a by-product, we note the following interesting consequence of Lemma 2.1.

COROLLARY 2.2. *Assume that in \mathbf{V} , there is no proper class inner model with a Woodin cardinal. Let R, R' be proper class extender models and $\sigma : R \rightarrow R'$ be an elementary map. Then R is normally iterable if and only if R' is normally iterable. Here “normally iterable” means that there is a uniqueness normal iteration strategy defined on all set-sized normal iteration trees.*

PROOF. Assume R is iterable in \mathbf{V} . Applying Lemma 2.1 to the situation where $\mathbf{M} = R$, we conclude that R is internally normally iterable, that is, R satisfies the statement “ R is normally iterable”. By the elementarity of the map σ , the model R' is internally normally iterable. Applying Lemma 2.1 again, this time with $\mathbf{M} = R'$, we conclude that R' is normally iterable in the sense of \mathbf{V} . \dashv

Recall that an extender model R is *fully iterable* if and only if R is iterable with respect to linear compositions of arbitrary (hence even class-sized) normal iteration trees. When working in a universe below the measurable cardinal Ω , normal/full iterability via the uniqueness strategy with respect to set-sized trees automatically guarantees normal/full iterability via the uniqueness strategy with respect to class-sized trees.

Working in the universe below Ω , fix a stationary class of ordinals that will serve as a reference class for the notion of *thickness*. In our applications we will assume for simplicity that this class consists of inaccessibles, so we let this class be the class A_0 as defined at the end of Steel [21, Section 1]. Then Ω is A_0 -thick in \mathbf{K}^c .

COROLLARY 2.3. *Assume that in \mathbf{V} , there is no proper class inner model with a Woodin cardinal. Let R be a proper class extender model that is fully iterable and such that Ω is A_0 -thick in R , and let F be an R -extender such that $\text{Ult}(R, F)$ is well-founded. Denote the transitive collapse of $\text{Ult}(R, F)$ by R' . Then R' is fully iterable.*

PROOF. By Corollary 2.2 and the discussion immediately following, R' is normally iterable with respect to class-sized normal iteration trees. It follows that there is an extender model R^* that is a common normal iterate of both R and R' . Since Ω is A_0 -thick in R , the corresponding normal iteration tree on R' does not involve any truncation on its main branch, and there is an iteration map $\pi : R' \rightarrow R^*$ that is fully elementary. Now R^* is fully iterable, as it is an iterate of R . The existence of π then guarantees that R' is fully iterable as well. \dashv

Finally we will need a well-known lemma on the absorption of extenders by the core model; this criterion is similar to Theorem 8.6 in [21]. Although the proof in [21] is inductive, our assumption allow us to avoid this kind of induction. Additionally, there is no need to impose any variant of the initial segment condition on the extender we intend to absorb by \mathbf{K} .

LEMMA 2.4. *Assume there is no proper class model with a Woodin cardinal. Let F be a (κ, λ) -extender over \mathbf{K} where λ is a \mathbf{K} -cardinal, and let W be a proper class extender model witnessing the A_0 -soundness of $\mathbf{K} \parallel \delta$ where δ is a cardinal larger than λ . Assume further that the following hold.*

- (a) *The phalanx $\langle W, W', \lambda \rangle$ is normally iterable where $W' = \text{Ult}(W, F)$.*
- (b) *$E^W \upharpoonright v = E^{W'} \upharpoonright v$ where $v = \lambda^{+W'}$.*

Then $F \in W$.

PROOF. We give a sketch of the proof. Compare W against the phalanx $\langle W, W', \lambda \rangle$. Let \mathcal{T}, \mathcal{U} be the iteration trees arising in the comparison where \mathcal{T} is on W and \mathcal{U}

is on the phalanx. Using the fact that Ω is A_0 -thick in W we conclude that \mathcal{T}, \mathcal{U} have a common last model W^* , the model W' is on the main branch of \mathcal{U} and the iteration maps $i^{\mathcal{T}} : W \rightarrow W^*$ and $i^{\mathcal{U}} : W' \rightarrow W^*$ exist. The critical point of $i^{\mathcal{U}}$ is $\geq \lambda$. Using the fact that W witnesses the A_0 -soundness of $\mathbf{K} \parallel \delta$ we conclude that $\text{cr}(i^{\mathcal{T}}) = \kappa$ and

$$i^{\mathcal{T}}(x) = i_{\mathcal{U}} \circ i_F(x) \text{ whenever } x \in \mathcal{P}(\kappa) \cap W, \tag{3}$$

where i_F is the ultrapower map associated with $\text{Ult}(W, F)$. By assumption (b) the first extender $E_{\alpha}^{\mathcal{T}}$ used on the main branch of \mathcal{T} has length at least ν , and since λ is a W -cardinal, the natural length of $E_{\alpha}^{\mathcal{T}}$ is at least λ . (Notice that here the fact that λ is a W -cardinal is only used to cover the case where $\alpha = 0$.) But then it follows from (3) combined with the fact that $\text{cr}(i_{\alpha+1, \infty}^{\mathcal{T}}), \text{cr}(i^{\mathcal{U}}) \geq \lambda$ that F is an initial segment of $E_{\alpha}^{\mathcal{T}}$. By applying the initial segment condition for $E_{\alpha}^{\mathcal{T}}$ along with the fact that $\text{lh}(E_0^{\mathcal{T}})$ is a cardinal in $\mathcal{M}_{\alpha}^{\mathcal{T}}$ if $\alpha > 0$ we conclude $F \in W$. \dashv

In general, if F is a (κ, λ) extender that satisfies condition (b) above, we say that F *coheres* to W .

§3. Universality.

3.1. Outline. Throughout this section we will work with a fixed proper class inner model \mathbf{M} and a cardinal δ which is regular and uncountable in \mathbf{V} . In order to guarantee that the core model theory is applicable, we will additionally assume that in \mathbf{V} there is no proper class inner model with a Woodin cardinal.

Our argument is based on a combination of ideas coming from the proof of universality of \mathbf{K}^c described in Jensen [11] and, in the simpler form reformulated for measures of order zero, also in Zeman [28, Section 6.4], further from the proof of universality of \mathbf{K}^c in Mitchell–Schindler [14], and finally from the Mitchell–Schimmerling–Steel proof of universality of $\mathbf{K} \parallel \delta$, see Schimmerling–Steel [18]. But of course, since we work under significantly restricted circumstances, we need to do some amount of extra work.

As it is usual in arguments of this kind, we will replace $\mathbf{K}^{\mathbf{M}}$, the true core model in \mathbf{M} , with a proper class fully iterable extender model W that in \mathbf{M} witnesses the A_0 -soundness of $\mathbf{K}^{\mathbf{M}} \parallel \delta'$ for a sufficiently large $\delta' > \delta$. This means that, working in \mathbf{M} , the model W extends $\mathbf{K}^{\mathbf{M}} \parallel \delta'$, is A_0 -thick, and $\delta' \subseteq H_1^W(\Gamma)$ whenever Γ is A_0 -thick in W ; here $H_1^W(\Gamma)$ denotes the Σ_1 -hull of Γ in W . (See Steel [21, Section 3] for details.) In our case, taking $\delta' = \delta^+$ will suffice. We recall that we are using Mitchell–Steel indexing of extenders, as this setting is convenient when referring to Steel [21].

Our general strategy is the following:

- (A) Assuming that in \mathbf{V} , there is an iterable premouse of size less than δ that iterates past $W \parallel \delta$, we construct a W -extender with critical point κ and support λ such that $F \in \mathbf{M}$ and coheres to W . (This does not require that $\text{Ult}(W, F)$ is well-founded.)
- (B) We prove that $\text{Ult}(W, F)$ is well-founded and, letting W' be its transitive collapse, prove that $\lambda^{+W} = \lambda^{+W'}$ and the phalanx (W, W', λ) is normally iterable in the sense of \mathbf{M} . We then apply Lemma 2.4 inside \mathbf{M} to conclude $F \in W$. This will yield a contradiction, as F codes a function which collapses λ^{+W} .

3.2. Part (A). We begin with (A). The construction in (A) is the same for both parts of Theorem 1.6; as mentioned above, we will use the Mitchell–Schindler approach to the proof of universality. Assume that

$$S \subseteq \mathcal{P}_\delta(\delta^+) \cap \mathbf{M} \tag{4}$$

is stationary in the sense of \mathbf{V} .

By Lemma 2.1, the model W is normally iterable via the uniqueness strategy in the sense of \mathbf{V} , so in particular it is coiterable with any iterable premouse in \mathbf{V} . Assume N is an iterable premouse in \mathbf{V} of size less than δ with an iteration strategy Σ that iterates past $\mathbf{K}^M \parallel \delta$. This means that there is a pair $(\mathcal{T}, \mathcal{U})$ of normal iteration trees arising in the coiteration of W against N , where \mathcal{T} is on W and \mathcal{U} is on N , \mathcal{T} is according to the uniqueness strategy for W , \mathcal{U} is according to Σ and the length of \mathcal{U} is $\delta + 1$. We view \mathcal{T}, \mathcal{U} as unpadding trees, so the length of \mathcal{T} is $\leq \delta + 1$. Also, we view \mathcal{T} as an iteration tree on $W \parallel \delta^+$.

Fix the following notation:

- W' is the last model of \mathcal{T} and N' is the last model of \mathcal{U} .
- The notation $\kappa_i^{\mathcal{T}}, \pi_{ij}^{\mathcal{T}}$ is self-explanatory, and $v_i^{\mathcal{T}}$ is the iteration index associated with the v_i -th extender on \mathcal{T} , that is, $v_i^{\mathcal{T}} = \text{lh}(E_i^{\mathcal{T}})$. We of course fix the corresponding notation for \mathcal{U} .
- $W_i = M_i^{\mathcal{T}}$ and $N_i = M_i^{\mathcal{U}}$.
- T , resp. U , is the tree structure associated with the iteration tree \mathcal{T} , resp. \mathcal{U} , and $<_T, <_U$ are the respective tree orderings.
- $b^{\mathcal{T}}$ is the main branch of \mathcal{T} and $b^{\mathcal{U}}$ is the main branch of \mathcal{U} . Note that $\delta \in b^{\mathcal{U}}$.
- For any $i < \delta$ we write $\xi^{\mathcal{T}}(i)$, resp. $\xi^{\mathcal{U}}(i)$, to denote the immediate $<_T$ -, resp. $<_U$ -, predecessor of $i + 1$.

LEMMA 3.1. *Under the above setting, the following hold:*

- (a) *There is no truncation point on $b^{\mathcal{T}}$.*
- (b) *For every $\alpha < \delta$ there is $i + 1 \in b^{\mathcal{T}}$ such that $\pi_{0, \xi(i)}^{\mathcal{T}}(\alpha) < \kappa_i^{\mathcal{T}} = \text{cr}(\pi_{\xi(i), b^{\mathcal{T}}}^{\mathcal{T}})$. Equivalently, $\pi_{b^{\mathcal{T}}}^{\mathcal{T}}(\delta) = \delta$.*

PROOF. Both clauses follow by standard arguments similar to those used in the proof of the termination of the comparison process and in the proof of universality. ⊖

Using a pressing down argument, we obtain a club $C^* \subseteq b^{\mathcal{U}}$ which is a thread through the set of critical points. Precisely, $\pi_{\xi, \xi'}^{\mathcal{U}}(\kappa_i^{\mathcal{U}}) = \kappa_j^{\mathcal{U}}$ whenever $\xi < \xi'$ are elements of C^* such that $\xi = \xi^{\mathcal{U}}(i)$ and $\xi' = \xi^{\mathcal{U}}(j)$. Of course $\min(C^*) \geq \xi^{\mathcal{U}}(i^{\mathcal{U}})$ where $i^{\mathcal{U}} + 1$ is the last truncation point on $b^{\mathcal{U}}$. We will additionally assume that elements of C^* satisfy the following:

- (i) $\xi = \kappa_i^{\mathcal{U}}$ whenever $\xi \in C^*$ and i is such that $\xi = \xi^{\mathcal{U}}(i)$, hence $\xi = \text{cr}(\pi_{\xi, \xi'}^{\mathcal{U}})$ for all $\xi' \in b^{\mathcal{U}} - (\xi + 1)$.
- (ii) $\pi_{0, \xi}^{\mathcal{T}}(\xi) = \xi$.
- (iii) ξ is a limit point of the critical points on $b^{\mathcal{T}}$ if $\text{lh}(\mathcal{T}) = \delta + 1$, and $\xi > \text{lh}(E_\alpha^{\mathcal{T}})$ for all $\alpha < \text{lh}(\mathcal{T})$ if $\text{lh}(\mathcal{T}) < \delta$.

That these requirements may be imposed without loss of generality follows from the fact that they are true on a club. For (i) this is obvious, and for (ii) and (iii)

we appeal to the proof of Lemma 3.1. Notice also that each element of C^* is an inaccessible cardinal in N' , and since N' agrees with W' below δ , also in W' .

LEMMA 3.2. *Given any $\xi \in C^*$, let $\tau_\xi = \xi^{+W}$, and $\tau'_\xi = \xi^{+W_\xi}$ if $\text{lh}(\mathcal{T}) = \delta + 1$ and $\tau'_\xi = \xi^{+W'}$ if $\text{lh}(\mathcal{T}) < \delta$. Then for every $\xi \in C^*$ the following is true:*

- (a) ξ is inaccessible in W_i for all $i \in b^\mathcal{T}$.
- (b) $\tau'_\xi = \xi^{+N_\xi} = \xi^{+N'}$.

PROOF. We do the proof for the case where $\text{lh}(\mathcal{T}) = \delta + 1$. If $\text{lh}(\mathcal{T}) < \delta$ the proof is similar but easier. Let $\xi \in C^*$ and i be such that $\xi = \xi^{\mathcal{T}}(i)$. Since the critical points on $b^\mathcal{T}$ are cofinal in ξ , necessarily $\kappa_i^\mathcal{T} \geq \xi$. The critical point of the map $\pi_{i+1,\delta}^\mathcal{T}$ is above ξ , and ξ is inaccessible in W' by the discussion immediately above the statement of the lemma, so ξ is an inaccessible cardinal in W_{i+1} . Since $W_{i+1} = \text{Ult}(W_\xi, E_i^\mathcal{T})$ and $E_i^\mathcal{T}$ is an extender with critical point at least ξ , the ordinal ξ is inaccessible in W_ξ . Since $\pi_{0,\xi}^\mathcal{T}(\xi) = \xi$, the ordinal ξ is inaccessible in all models W_i for $i <_\mathcal{T} \xi$. This proves (a). Regarding (b), we have $\mathcal{P}(\xi) \cap W_{\xi^{\mathcal{T}}(i)} = \mathcal{P}(\xi) \cap W_{i+1}$, by our choice of ξ and general properties of iterations trees. Since $\text{cr}(\pi_{i+1,\delta}^\mathcal{T}) > \xi$, we conclude $\tau'_\xi = \xi^{+W'} = \xi^{+N'} = \xi^{+N_\xi}$, where the last equality follows by the same considerations for \mathcal{U} as have been just done for \mathcal{T} . \dashv

The initial setting recorded above is more or less the same as in Jensen’s proof of universality of \mathbf{K}^c in Jensen [11]. Although the next two lemmata follow the general scenario of his proof, the arguments we use are significantly more local, and we need to do this for two reasons. First, we work in \mathbf{V} but on the other hand, the model whose universality we want to establish is in \mathbf{M} , which restricts our freedom in choosing the structures to work with. Second, we work below a regular cardinal δ given in advance which—unlike the situation in Jensen’s proof—we would like to be as small as possible, and therefore it need not have any reasonable closure properties. We stress that we are not making any assumptions on cardinal arithmetic. These two lemmata comprise one of the key observations in our argument.

LEMMA 3.3. *Let $\xi \in \lim(C^*)$ and $\bar{\xi} < \zeta < \tau_\xi$. Then there is a $\xi^* \in C^*$ such that $\xi^* < \xi$ and, for all $\bar{\xi} \in C^*$ with $\xi^* \leq \bar{\xi} < \xi$, we have*

$$(\pi_{\bar{\xi},\xi}^\mathcal{U})^{-1} \circ \pi_{0,\xi}^\mathcal{T}(\zeta) \in \text{rng}(\pi_{0,\bar{\xi}}^\mathcal{T}).$$

PROOF. Since $\bar{\xi} < \zeta < \tau_\xi$, there is a set $a \in W$ such that $a \subseteq \bar{\xi} \times \xi$ and a is a well-ordering of $\bar{\xi}$ of order-type ζ . Let ξ^* be the least element ξ' of C^* such that $\pi_{0,\xi}^\mathcal{T}(a) \in \text{rng}(\pi_{\xi',\xi}^\mathcal{U})$. Then for $\bar{\xi} \in [\xi^*, \xi) \cap C^*$ we have

$$\pi_{0,\xi}^\mathcal{T}(a) = \pi_{\bar{\xi},\xi}^\mathcal{U}(\pi_{0,\xi}^\mathcal{T}(a) \cap \bar{\xi}) = \pi_{\bar{\xi},\xi}^\mathcal{U} \circ \pi_{0,\bar{\xi}}^\mathcal{T}(a \cap \bar{\xi}).$$

Here the first equality follows from the fact that $\bar{\xi} = \text{cr}(\pi_{\bar{\xi},\xi}^\mathcal{U})$ and $\pi_{\bar{\xi},\xi}^\mathcal{U}(\bar{\xi}) = \bar{\xi}$. To see the second equality notice that $\text{cr}(\pi_{\bar{\xi},\xi}^\mathcal{T}) \geq \bar{\xi}$, so $\pi_{0,\xi}^\mathcal{T}(a) \cap \bar{\xi} = \pi_{0,\bar{\xi}}^\mathcal{T}(a)$. By the elementarity of all relevant maps, $a \cap \bar{\xi}$ is a well-ordering of $\bar{\xi}$. Letting $\bar{\zeta}$ be the order-type of $a \cap \bar{\xi}$, we conclude that $(\pi_{\bar{\xi},\xi}^\mathcal{U})^{-1} \circ \pi_{0,\xi}^\mathcal{T}(\zeta) = \pi_{0,\bar{\xi}}^\mathcal{T}(\bar{\zeta})$. \dashv

Given a map f and a set x , we will write $f[x]$ to denote the pointwise image of x under f .

LEMMA 3.4. *There is a $\xi^* \in C^*$ such that for every $\xi, \xi' \in C^*$ with $\xi^* \leq \xi < \xi'$ we have $\pi_{\xi, \xi'}^U \circ \pi_{0, \xi}^T[W \parallel \tau_\xi] \subseteq \pi_{0, \xi'}^T[W \parallel \tau_{\xi'}]$. Moreover, the set on the left is a cofinal subset of the set on the right.*

PROOF. To prove the inclusion, it suffices to show that $\pi_{\xi, \xi'}^U \circ \pi_{0, \xi}^T[\tau_\xi] \subseteq \pi_{0, \xi'}^T[\tau_{\xi'}]$ for all $\xi < \xi'$ on a tail-end of C^* . The first step toward the proof of this inclusion is the following claim. Let $\gamma = \min(C^*)$.

CLAIM 3.5. *Let $\gamma < \beta < \tau'_\gamma$ (see Lemma 3.2 for the definition of τ'_γ). There is a $\xi_\beta \in C^*$ and a sequence*

$$\langle \xi_\xi \mid \xi \in C^* - \xi_\beta \rangle$$

such that the following holds:

- (a) $\xi < \xi_\xi < \tau_\xi$ and $\pi_{0, \xi}^T(\xi_\xi) \geq \pi_{\gamma, \xi}^U(\beta)$ whenever $\xi \in C^* - \xi_\beta$, and
- (b) $\pi_{\xi, \xi'}^U \circ \pi_{0, \xi}^T(\xi_\xi) = \pi_{0, \xi'}^T(\xi_{\xi'})$ whenever $\xi < \xi'$ are in $C^* - \xi_\beta$.

PROOF. Pick a sequence $\langle \bar{\xi}_\xi \mid \xi \in \text{lim}(C^*) \rangle$ such that $\xi < \bar{\xi}_\xi < \tau_\xi$ and $\pi_{0, \delta}^T(\bar{\xi}_\xi) \geq \pi_{\gamma, \xi}^U(\beta)$ for all $\xi \in \text{lim}(C^*)$. Using Lemma 3.3 we obtain a regressive function g defined on $\text{lim}(C^*)$ such that $(\pi_{\xi, \xi}^U)^{-1} \circ \pi_{0, \xi}^T(\bar{\xi}_\xi) \in \text{rng}(\pi_{0, \xi}^T)$ whenever $\xi \in \text{lim}(C^*)$ and $\bar{\xi} \in [g(\xi), \xi] \cap C^*$. Using Fodor's lemma, we find a $\xi_\beta \in C^*$ and a stationary set $A'_\beta \subseteq \text{lim}(C^*)$ such that $g(\xi) = \xi_\beta$ for all $\xi \in A'_\beta$. Then, using the pigeon-hole principle along with the fact that $\tau'_{\xi_\beta} < \delta$, we find an ordinal ξ'_β such that $\xi_\beta < \xi'_\beta < \tau'_{\xi_\beta}$, and a stationary set $A_\beta \subseteq A'_\beta$ such that $\pi_{\xi_\beta, \xi}^U(\xi'_\beta) = \pi_{0, \xi}^T(\bar{\xi}_\xi)$ for all $\xi \in A_\beta$. For $\xi \in C^* - \xi_\beta$ then let

$$\xi_\xi = (\pi_{0, \xi}^T)^{-1} \circ \pi_{\xi_\beta, \xi}^U(\xi'_\beta).$$

Notice that ξ_ξ is defined for every $\xi \in C^* - \xi_\beta$: Just pick any $\xi' \in A_\beta$ such that $\xi' \geq \xi$; then

$$\pi_{\xi, \xi'}^U(\xi'_\beta) = (\pi_{\xi, \xi'}^U)^{-1} \circ \pi_{0, \xi'}^T(\bar{\xi}_{\xi'})$$

and the right side is in the range of $\pi_{0, \xi}^T$ by Lemma 3.3. (Notice that for $\xi \in A_\beta$ we have $\bar{\xi}_\xi = \bar{\xi}_{\xi}$.) Clause (a) of the Claim then follows easily from our choice of ordinals $\bar{\xi}_\xi$. Now if $\xi < \xi'$ are in $C^* - \xi_\beta$, then

$$\pi_{\xi, \xi'}^U \circ \pi_{0, \xi}^T(\xi_\xi) = \pi_{\xi, \xi'}^U \circ \pi_{\xi_\beta, \xi}^U(\xi'_\beta) = \pi_{\xi_\beta, \xi'}^U(\xi'_\beta) = \pi_{0, \xi'}^T(\bar{\xi}_{\xi'}),$$

which verifies (b) of the Claim. ◻

We now complete the proof of Lemma 3.4. Let $\varepsilon = \text{cf}^V(\tau'_\gamma)$. Pick a sequence $\langle \beta_i \mid i < \varepsilon \rangle$ that is increasing and cofinal in τ'_γ . Appealing to the above Claim, for each $i < \varepsilon$ there is $\xi^i = \xi_{\beta_i}$ and a sequence $\langle \xi^i_\xi \mid \xi \in C^* - \xi^i \rangle$ such that (a) and (b) of the Claim hold with ξ^i, ξ^i_ξ in place of ξ_β, ξ_ξ . Let $\xi^* \in C^*$ be an upper bound for the ξ^i ; such a ξ^* exists as $\varepsilon < \delta$. Then (a) and (b) of the above Claim hold with ξ^*, ξ^i_ξ in place of ξ_β, ξ_ξ whenever $i < \varepsilon$. Also, since each $\pi_{0, \xi}^T$ maps τ_ξ cofinally into τ'_ξ , and $\pi_{\xi, \xi}^U$ maps τ'_ξ cofinally into τ'_ξ , each sequence $\langle \xi^i_\xi \mid i < \varepsilon \rangle$ is cofinal in τ_ξ . It follows that for any $\xi < \xi'$ in $C^* - \xi^*$, the assignment $\xi^i_\xi \mapsto \xi^i_{\xi'}$ maps a cofinal subset of τ_ξ cofinally into $\tau_{\xi'}$.

Fix $\xi < \xi'$ in $C^* - \xi^*$ and an $i < \varepsilon$. Let $f, f' \in W$ be such that f is the $<_{EW}$ -least surjection of ξ onto $M \parallel \xi^i_\xi$ and f' is the $<_{EW}$ -least surjection of ξ' onto $W \parallel \xi^i_{\xi'}$;

recall that E^W is the extender sequence of W . Then $f \in W \parallel \tau_\xi$ and $f' \in W \parallel \tau_{\xi'}$. Appealing to (b) of the Claim and to the elementarity of the maps $\pi_{0,\xi}^T$ and $\pi_{\xi,\xi'}^U$, we conclude that $\pi_{\xi,\xi'}^U \circ \pi_{0,\xi}^T(f) = \pi_{0,\xi'}^T(f')$. Given $x \in W \parallel \zeta_\xi^i$, let $\eta < \xi$ be such that $x = f(\eta)$. Then

$$\begin{aligned} \pi_{\xi,\xi'}^U \circ \pi_{0,\xi}^T(x) &= \pi_{\xi,\xi'}^U \circ \pi_{0,\xi}^T(f(\eta)) = \pi_{\xi,\xi'}^U \circ \pi_{0,\xi}^T(f) (\pi_{\xi,\xi'}^U \circ \pi_{0,\xi}^T(\eta)) \\ &= \pi_{0,\xi'}^T(f')(\pi_{0,\xi}^T(\eta)) = \pi_{0,\xi'}^T(f') ((\pi_{0,\xi'}^T(\eta))) = \pi_{0,\xi'}^T(f'(\xi)) \\ &= \pi_{0,\xi'}^T(x). \end{aligned}$$

Here the first two equalities on the middle line follow from the fact that $cr(\pi_{\xi,\xi'}^U) = \xi$ and $cr(\pi_{0,\xi'}^T) \geq \xi$.

Since the sequence $\langle \zeta_\xi^i \mid i < \gamma \rangle$ is cofinal in τ_ξ , the computation in the previous paragraph yields $\pi_{\xi,\xi'}^U \circ \pi_{0,\xi}^T[W \parallel \tau_\xi] \subseteq \pi_{0,\xi'}^T[W \parallel \tau_{\xi'}]$. As the assignment $\zeta_\xi^i \mapsto \zeta_{\xi'}^i$ is cofinal in $\tau_{\xi'}$ (as we saw above), it follows that the set on the left is a cofinal subset of the set on the right. This completes the proof of Lemma 3.4. \dashv

Without loss of generality we will assume that the conclusions of Lemma 3.4 hold for any pair of ordinals $\xi < \xi'$ in C^* . For any such pair of ordinals we let

$$\pi_{\xi,\xi'} = (\pi_{0,\xi'}^T)^{-1} \circ \pi_{\xi,\xi'}^U \circ \pi_{0,\xi}^T \upharpoonright (W \parallel \tau_\xi).$$

Then $\pi_{\xi,\xi'} : W \parallel \tau_\xi \rightarrow W \parallel \tau_{\xi'}$ is a fully elementary map such that

- $cr(\pi_{\xi,\xi'}) = \xi$ and $\pi_{\xi,\xi'}(\xi) = \xi'$, and
- $\pi_{\xi,\xi'}$ maps $W \parallel \tau_\xi$ cofinally into $W \parallel \tau_{\xi'}$.
- The system $\langle W \parallel \tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^* \rangle$ is commutative and continuous, so if ξ is a limit point of C^* , then $W \parallel \tau_\xi$ is the direct limit of the system $\langle W \parallel \tau_{\bar{\xi}}, \pi_{\bar{\xi},\xi'} \mid \bar{\xi} < \xi' \text{ in } C^* \cap \xi \rangle$.

The above conclusions follow easily from the definition of $\pi_{\xi,\xi'}$ and the previous calculations. We give some details on the continuity: Given a limit point ξ of C^* , pick some $\bar{\xi} \in C^* \cap \xi$. If $x \in W \parallel \tau_\xi$, pick some $\zeta < \tau_{\bar{\xi}}$ such that $x \in W \parallel \pi_{\bar{\xi},\xi}(\zeta)$; this is possible by the cofinality of the map $\pi_{\bar{\xi},\xi}$. If $\bar{f} \in W \parallel \tau_{\bar{\xi}}$ is a surjection of $\bar{\xi}$ onto $W \parallel \zeta$, then $f = \pi_{\bar{\xi},\xi}(\bar{f})$ is a surjection of ξ onto $W \parallel \pi_{\bar{\xi},\xi}(\zeta)$, so $x = f(\alpha)$ for some $\alpha < \xi$. Now pick $\xi' \in C^* \cap [\bar{\xi}, \xi)$ such that $\alpha < \xi'$. Since f and α belong to $\text{rng}(\pi_{\xi',\xi})$, it follows that $x \in \text{rng}(\pi_{\xi',\xi})$ as well.

LEMMA 3.6. *Let $\xi < \xi'$ be in C^* . Let $F_{\xi,\xi'}$ be the (ξ, ξ') -extender derived from $\pi_{\xi,\xi'}$. Then $F_{\xi,\xi'}$ is a W -extender that coheres to W .*

PROOF. We verify that $F_{\xi,\xi'}$ coheres to W ; that $F_{\xi,\xi'}$ is a W -extender is clear. As $W \parallel \tau_\xi \models \text{ZFC}^-$, the ultrapower $\text{Ult}(W \parallel \tau_\xi, F_{\xi,\xi'})$ can be elementarily embedded into $W \parallel \tau_{\xi'}$ in the usual way by assigning $\pi_{\xi,\xi'}(f)(a)$ to the object in the ultrapower represented by $[a, f]$. This shows that the ultrapower is well-founded. Let Q be its transitive collapse, and let $k : Q \rightarrow W \parallel \tau_{\xi'}$ be the embedding defined as above. We then have $\sigma \upharpoonright (\xi' + 1) = \text{id}$. It follows that $k = \text{id}$, as otherwise $cr(k)$ would be a cardinal in Q larger than ξ' . Hence Q is an initial segment of $W \parallel \tau_{\xi'}$. Since $\pi_{\xi,\xi'}$ maps τ_ξ cofinally into $\tau_{\xi'}$ and $\pi_{\xi,\xi'} = k \circ i$, where $i : Q \rightarrow W \parallel \tau_{\xi'}$ is the ultrapower embedding, then k maps $\text{On} \cap Q$ cofinally into $\tau_{\xi'}$. It follows that $Q = W \parallel \tau_{\xi'}$. \dashv

Let

$$\tilde{W} \text{ be the direct limit of the diagram } \langle W \parallel \tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^* \rangle, \tag{5}$$

and

$$\pi_\xi : W \parallel \tau_\xi \rightarrow \tilde{W} \text{ be the direct limit maps.} \tag{6}$$

LEMMA 3.7. *Let θ be a regular cardinal larger than δ^+ , and let X be an elementary substructure of H_θ such that $\kappa = X \cap \delta \in \delta$ and*

$$\langle W \parallel \tau_\xi, \pi_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \rangle \in X.$$

Let H be the transitive collapse of X , and let $\sigma : H \rightarrow H_\theta$ be the inverse of the Mostowski collapsing isomorphism. Then $\kappa \in C^$, $W \parallel \tau_\kappa \in H$, $\sigma(W \parallel \tau_\kappa) = \tilde{W}$, and $\sigma \upharpoonright (W \parallel \tau_\kappa) = \pi_\kappa$.*

PROOF. Since $C^* \in X$ and C^* is club in δ , the critical point κ of σ is a limit point of C^* , hence an element of C^* . Also $\sigma(C^* \cap \kappa) = C^*$. The inverse image of the diagram $\langle W \parallel \tau_\xi, \pi_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \rangle$ under σ is thus of the form $\langle \tilde{W}_\xi, \tilde{\pi}_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \cap \kappa \rangle$ where $\sigma(\tilde{W}_\xi) = W \parallel \tau_\xi$ for all $\xi \in C^* \cap \kappa$. Since the height of each \tilde{W}_ξ is strictly smaller than κ , the map σ is the identity on \tilde{W}_ξ , so $\tilde{W}_\xi = W \parallel \tau_\xi$ and $\tilde{\pi}_{\xi, \xi'} = \pi_{\xi, \xi'}$ for all $\xi < \xi' \text{ in } C^* \cap \kappa$. Because the diagram $\langle W \parallel \tau_\xi, \pi_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \rangle$ is continuous, the structure $W \parallel \tau_\kappa = \lim \langle W \parallel \tau_\xi, \pi_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \cap \kappa \rangle$, being a direct limit of a diagram in H is itself an element of H , and the same is also true of the direct limit maps $\pi_{\xi, \kappa}$ where $\xi \in C^* \cap \kappa$. It follows that

$$\begin{aligned} \sigma(W \parallel \tau_\kappa) &= \sigma(\lim \langle W \parallel \tau_\xi, \pi_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \cap \kappa \rangle) \\ &= \lim \langle W \parallel \tau_\xi, \pi_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \rangle = \tilde{W}. \end{aligned}$$

Furthermore, from the elementarity of σ , we obtain that $\sigma(\pi_{\xi, \kappa}) = \pi_\xi$ for all ordinals $\xi \in C^* \cap \kappa$. Thus if $x \in W \parallel \tau_\kappa$, then $x = \pi_{\xi, \kappa}(\bar{x})$ for some $\xi \in C^* \cap \kappa$ and $\bar{x} \in W \parallel \tau_\xi$, hence

$$\sigma(x) = \sigma(\pi_{\xi, \kappa}(\bar{x})) = \sigma(\pi_{\xi, \kappa})(\sigma(\bar{x})) = \pi_{\xi, \delta}(\bar{x}) = \pi_{\kappa, \delta}(\pi_{\xi, \kappa}(\bar{x})) = \pi_{\kappa, \delta}(x),$$

which shows that $\sigma \upharpoonright (W \parallel \tau_\kappa) = \pi_\kappa$. ⊖

It follows from the above lemma that the collection of all elementary substructures X of H_θ such that $\kappa = X \cap \delta \in \delta$ and \tilde{W} (from (5)) collapses to $W \parallel \kappa^{+W}$ under the Mostowski collapsing isomorphism coming from X , is a club subset of $\mathcal{P}_\delta(H_\theta)$. This is a strengthening of Mitchell–Schindler [14, Lemma 3.5].

We do not know if there is a significantly simpler proof of Lemma 3.7 based on the idea used in Mitchell–Schindler [14, Lemma 3.5], as it seems that one needs to require $\tilde{W} \in X$ in order to obtain the conclusions of Lemma 3.7.

Given a regular cardinal $\theta > \delta^+$ and an elementary substructure X of H_θ such that $X \cap \delta \in \delta$ and the diagram $\langle W \parallel \tau_\xi, \pi_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \rangle$ is an element of X , define the following objects:

- H_X is the transitive collapse of X and $\sigma_X : H_X \rightarrow X$ is the inverse of the Mostowski collapsing isomorphism.
- $\kappa_X = \text{cr}(\sigma_X)$ and $\tau_X = \tau_{\kappa_X} = \kappa_X^{+W}$.

If Y is another such structure, $Y \supseteq X$ and $\kappa_Y > \kappa_X$ then:

- $\sigma_{X,Y} : H_X \rightarrow H_Y$ is defined by $\sigma_{X,Y} = \sigma_Y^{-1} \circ \sigma_X$.
- We write π_X for π_{κ_X} and $\pi_{X,Y}$ for π_{κ_X, κ_Y} .

Here of course all objects introduced above depend on θ , but in our applications we will keep θ fixed, so we treat it as a suppressed parameter. It follows from the

above lemma that $\sigma_X \upharpoonright (W \parallel \tau_X) = \pi_X$ and $\sigma_{X,Y} \upharpoonright (W \parallel \tau_X) = \pi_{X,Y}$, so these restrictions depend only on the ordinals κ_X and κ_Y and not on the structures X and Y themselves. In particular we get

$$F_{X,Y} \stackrel{\text{def}}{=} F_{\kappa_X, \kappa_Y} = \text{the } W\text{-extender at } (\kappa_X, \kappa_Y) \text{ derived from } \sigma_{X,Y}, \tag{7}$$

and

$$F_X \stackrel{\text{def}}{=} \text{the } W\text{-extender at } (\kappa, \delta) \text{ derived from } \pi_X = \sigma_X \upharpoonright (W \parallel \tau_X). \tag{8}$$

We also note that if $\theta > \delta$ then $\mathcal{P}_\delta(H_\theta)$ is defined consistently with our definition of $\mathcal{P}_\kappa(\lambda)$, that is, elements X of $\mathcal{P}_\delta(H_\theta)$ have the property $X \cap \delta \in \delta$.

LEMMA 3.8. *Given a stationary set S as in (4), let S^θ be the collection of all $X \in \mathcal{P}_\delta(H_\theta)$ satisfying the following requirements:*

- (a) X is an elementary substructure of H_θ .
- (b) The diagram $\langle W \parallel \tau_\xi, \pi_{\xi, \xi'} \mid \xi < \xi' \text{ in } C^* \rangle$ is an element of X .
- (c) $X \cap \delta^+ \in S$.

Then S^θ is stationary, and for any $X, Y \in S^\theta$ such that $X \subseteq Y$ and $\kappa_X < \kappa_Y$, the extender $F_{X,Y}$ is an element of \mathbf{M} .

PROOF. The stationarity of S^θ follows from the stationarity of S and from the fact that the collection of all X satisfying clauses (a) and (b) is a club in $\mathcal{P}_\delta(H_\theta)$. We now verify that $F_{X,Y} \in \mathbf{M}$. For this, it will suffice to see that $\pi_{X,Y} \in \mathbf{M}$. We first observe that $\pi_{X,Y} \upharpoonright \tau_X \in \mathbf{M}$; for this we use the fact that $\pi_{X,Y} \upharpoonright \tau_X = (\pi_Y^{-1} \upharpoonright \tau_Y) \circ (\pi_X \upharpoonright \tau_X)$. To see that both restriction maps on the right of this equality are in \mathbf{M} , let us argue for instance for X . Since $X \in S^\theta$ and $S \subseteq \mathbf{M}$, the restriction $\pi_X \upharpoonright \text{otp}(X \cap \delta^+) = \sigma_X \upharpoonright \text{otp}(X \cap \delta^+)$, being the (unique) isomorphism between $\text{otp}(X \cap \delta^+)$ and $X \cap \delta^+$, is an element of \mathbf{M} . The equality here comes from Lemma 3.7. As $\tau_X \leq \text{otp}(X \cap \delta^+)$, the restriction $\pi_X \upharpoonright \tau_X$ is an element of \mathbf{M} as well.

Now, since $W \parallel \tau_X$ and $W \parallel \tau_Y$ are both in \mathbf{M} , it suffices to argue that $\pi_{X,Y}$ is fully determined by $\pi_{X,Y} \upharpoonright \tau_X$. Since $W \parallel \tau_X \models \text{ZFC}^-$, the canonical well-ordering $<_W$ orders $W \parallel \tau_X$ in order-type τ_X . So if $a \in W \parallel \tau_X$, then there is an ordinal $\alpha < \tau_X$ such that a is the α -th element of $W \parallel \tau_X$ under $<_W$. As $\pi_{X,Y}$ is elementary, $\pi_{X,Y}(a)$ is the $\pi_{X,Y}(\alpha)$ -th element of $W \parallel \tau_Y$ under $<_W$. So the value of $\pi_{X,Y}(a)$ is fully determined by the action of $\pi_{X,Y}$ on the elements of τ_X . \dashv

3.3. Proof of Theorem 1.6(a). So far we completed task (A) in the strategy outlined at the beginning of this section. We now proceed toward completing task (B). Our argument here depends on the anti-large cardinal hypothesis we make in \mathbf{V} . We start with the case where no proper class inner model with a Woodin cardinal in the sense of \mathbf{V} is allowed, that is, we are about to complete the proof of Theorem 1.6(a).

Here we apply a frequent extension argument, similar to that in Schimmerling–Steel [18]; see also Mitchell–Schimmerling [13]. A general version of the frequent extension argument in the simplified context of extender models below 0^\sharp is described in Zeman [28, Sections 7.5, 8.3], and also Cox [5]. The layout of the argument in Cox [5] uses a system of extender models indexed by structures from a suitable stationary set, rather than a linear chain of internally approachable structures. In our situation, it will be convenient to use this layout. The initial step of the frequent extension argument thus produces a system of structures and embeddings

indexed with elements X of the collection S_δ^θ for a suitable θ (see below for the definition of S_δ^θ); we denote these objects by $\bar{N}_X, N_X^*, \rho_X, \tilde{\sigma}_X, \sigma'_X, \dots$. We construct these objects in such a way that for each index X , each individual \bar{N}_X, N_X^*, \dots is an element of \mathbf{M} . The entire indexed system, of course, need not be an element of \mathbf{M} . It may be the case (we do not know) that one can proceed more liberally and run some constructions in \mathbf{V} instead of \mathbf{M} , but in the end we need to guarantee that certain premice and phalanxes we construct are in \mathbf{M} , and are iterable in the sense of \mathbf{M} . Our approach guarantees this automatically. Also, it is worth noting that in order to run the frequent extension argument, we merely need the antilarge cardinal hypothesis that no proper class inner model with a Woodin cardinal exists in the sense of \mathbf{M} . (So at this point in the proof of Theorem 1.6(a) we do not need the full antilarge cardinal assumption of the theorem.)

Recall the set S_δ from (2). Working in \mathbf{V} , pick a regular θ such that $H_{\delta^+} \in H_\theta$. Following the notation introduced in Lemma 3.8, set $S_\delta^\theta = (S_\delta)^\theta$. We show that for all but nonstationarily many $X \in S_\delta^\theta$ and all $Y \in S_\delta^\theta$ such that $X \in Y$, the ultrapower $\text{Ult}(W, F_{X,Y})$ is well-founded and the phalanx $(W, \text{Ult}(W, F_{X,Y}), \kappa_Y)$ is normally iterable in the sense of \mathbf{M} . (Under our antilarge cardinal hypothesis, this is actually equivalent to its iterability in the sense of \mathbf{V}). By Lemma 2.4 applied inside \mathbf{M} , the extender $F_{X,Y}$ is an element of W . However, by Lemma 3.6, the extender induces the ultrapower map $\pi_{X,Y}$ that maps τ_X cofinally into τ_Y , which is false in W . This is a contradiction, and completes the proof of Theorem 1.6(a).

REMARK 3.9. It is the use of the frequent extension argument that necessitates the assumption of stationarity of S_δ in place of the weaker assumption that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary. In general, the assumption that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M} - S_\delta$ is stationary is not sufficient to run the frequent extension argument; see for instance Räsch–Schindler [17].

Before we proceed, recall that given phalanxes (P, P', α) and (Q, Q', β) , a pair of maps (σ, σ') is an embedding of (P, P', α) into (Q, Q', β) if and only if $\sigma : P \rightarrow P'$ and $\sigma' : Q \rightarrow Q'$ are Σ_0 -elementary and cardinal preserving, $\sigma \upharpoonright \alpha = \sigma' \upharpoonright \alpha$, $\sigma[\alpha] \subseteq \beta$, and $\sigma'(\alpha) \geq \beta$. We will only deal with embeddings that are fully elementary, but of course one may consider embeddings of various degrees of preservation.

If \mathcal{T} is a normal iteration tree on (P, P', α) according to the uniqueness strategy, α is a cardinal in P' and (Q, Q', β) is iterable via some iteration strategy Σ^Q , we can copy \mathcal{T} onto a normal iteration tree on (Q, Q', β) according to Σ^Q via the embedding (σ, σ') . Note that we will use this terminology also in the case where $Q = Q'$, that is, when we talk about an embedding of a phalanx into a premouse.

For each $X \in S_\delta^\theta$, pick some $Y = Y(X) \in S_\delta^\theta$ such that $X \in Y$. Then let $G_X = F_{X,Y(X)}$ and $\lambda_X = \kappa_{Y(X)}$. We will prove that for all but nonstationarily many $X \in S_\delta^\theta$,

$$\text{Ult}(W, G_X) \text{ is well-founded and } (W, \text{Ult}(W, G_X), \lambda_X) \text{ is normally iterable.} \tag{9}$$

Here we talk about iterability in the sense of \mathbf{M} .

Heading for a contradiction, assume there is a stationary

$$S \subseteq S_\delta^\theta \tag{10}$$

such that either $\text{Ult}(W, G_X)$ is ill-founded or else this ultrapower is well-founded but the phalanx $(W, \text{Ult}(W, G_X), \lambda_X)$ is not normally iterable in the sense of \mathbf{M} . So there

is some $\theta_0 > \delta^{+V}$ such that θ_0 is a successor cardinal in W and the same is true with $W \parallel \theta_0$ in place of W for all $X \in S$. Write N for $W \parallel \theta_0$ and N'_X for $\text{Ult}(N, G_X)$, where we have identified well-founded parts with their transitive collapses. Thus even if N'_X is ill-founded, it is well-founded past λ_X , and if N'_X is well-founded, then it is actually transitive. For each $X \in S$ let $\mathcal{T}_X \in \mathbf{M}$ be a putative normal iteration tree on the phalanx (N, N'_X, λ_X) witnessing that the phalanx is not normally iterable in \mathbf{M} . Here we mean that either \mathcal{T}_X has a last ill-founded model, or \mathcal{T}_X is of limit length and has no cofinal well-founded branch. If N'_X is ill-founded, we let \mathcal{T}_X be the phalanx (N, N'_X, λ_X) whose last model is N'_X , which offers uniform treatment of the situations listed above.

Pick a \mathbf{V} -regular $\theta^* > \theta$ such that $\theta, S, \mathcal{T}^X \in H_{\theta^*}$ whenever $X \in S$. For each X , construct an elementary substructure $Z_X \prec H_{\theta^*}^{\mathbf{M}}$ such that

- $Z_X \in \mathbf{M}$ and is countable in \mathbf{M} , and
- $G_X, \mathcal{T}_X, \tilde{\tau} \in Z_X$, where $\tilde{\tau} = \delta^{+\tilde{W}}$.

Let H_X^Z be the transitive collapse of Z_X , let $\rho_X : H_X^Z \rightarrow H_{\theta^*}^{\mathbf{M}}$ be the inverse to the associated Mostowski collapsing isomorphism, and let

$$(\bar{\mathcal{T}}_X, \bar{N}_X, \bar{N}'_X, \bar{\lambda}_X) = \rho_X^{-1}(\mathcal{T}_X, N, N'_X, \lambda_X).$$

So each of these objects individually is in \mathbf{M} , although the collection consisting of all of them may not be an element of \mathbf{M} .

Next we do a pressing down argument which is a typical part of any frequent extension argument. Consider $X \in S$ such that $H_{\delta^+}^{\mathbf{M}} \in X$. Recall that since $X \cap \delta^+ \in \mathbf{M}$, the restriction $\sigma_X \upharpoonright \tau_X$ is an element of \mathbf{M} as well. Since δ is a cardinal and $\sigma_X \upharpoonright \tau_X$ maps $\tau_X < \delta$ cofinally into $\tilde{\tau}$, necessarily $\tilde{\tau} < \delta^{+\mathbf{M}}$, so there is a surjection $f : \delta \rightarrow \tilde{\tau}$ such that $f \in \mathbf{M}$. As $\tilde{\tau}, H_{\delta^+}^{\mathbf{M}} \in X$, there is a surjection $f : \delta \rightarrow \tilde{\tau}$ such that $f \in \mathbf{M} \cap X$. Since $\sigma_X[Z_X \cap \tau_X] \subseteq \mathbf{M} \cap X$, it follows that $\sigma_X[Z_X \cap \tau_X] \subseteq f[\kappa_X]$.

Since $\sigma_X \upharpoonright \tau_X$ is an element of \mathbf{M} , so is $\sigma_X[Z_X \cap \tau_X]$. In \mathbf{M} , the set $\sigma_X[Z_X \cap \tau_X]$ is a countable subset of $\tilde{\tau}$. Since $\text{cf}^{\mathbf{M}}(\kappa_X)$ is uncountable in \mathbf{M} , we can find an ordinal $\alpha < \kappa_X$ such that $\sigma_X[Z_X \cap \tau_X] \subseteq f[\alpha]$. But since $f \in X$, also $f[\alpha]$ is an element of X . Thus, letting $a = f[\alpha]$, the set X witnesses the existential quantifier in the statement

$$H_{\theta^*} \models (\exists v \in S)(a \in v). \tag{11}$$

Obviously the set

$$S' = \{X \in S \mid H_{\delta^+}^{\mathbf{M}} \in X \text{ and } (\exists X' \prec H_{\theta^*})(S \in X' \ \& \ X' \cap H_{\theta} = X)\}$$

is stationary. Given $X \in S'$, let $X' \prec H_{\theta^*}$ witness this. Since $a, S \in X'$, applying the elementarity of X' to (11) yields

$$X' \models (\exists v \in S)(a \in v).$$

So there is some $\bar{X} \in S \cap X'$ with $a \in \bar{X}$, and such \bar{X} is obviously an element of X . Since $\sigma_X[Z_X \cap \tau_X] \subseteq a \subseteq \bar{X}$ (the latter inclusion follows from the facts that $a \in \bar{X}$, the cardinality of a is less than δ , and $\bar{X} \cap \delta$ is transitive), we can define a regressive function $g : S' \rightarrow S$ such that $\sigma_X[Z_X \cap \tau_X] \subseteq g(X)$ for all $X \in S'$. By pressing down, there are a stationary $S^* \subseteq S'$ and an $X^* \in S$ such that

$$\sigma_X[Z_X \cap \tau_X] \subseteq X^* \in X \text{ for all } X \in S^*. \tag{12}$$

Fix the following notation:

- H^* is the transitive collapse of X^* .
- $\sigma_X^* : H_X^Z \rightarrow H^*$ is a partial map defined by $\sigma_X^* = \sigma_{X^*,X}^{-1} \circ \rho_X$.
- $\tau_X^* = \sup(\sigma_X^*[\bar{\tau}_X])$, where $\bar{\tau}_X = \rho_X^{-1}(\tau_X)$.

It follows from (12) that $\rho_X[\bar{\tau}_X] = Z_X \cap \tau_X \subseteq \sigma_{X^*,X}[\tau_{X^*}]$, so $\sigma_X^*(\xi)$ is defined for all $\xi < \bar{\tau}_X$. Furthermore, since $\rho_X, \sigma_{X^*,X} \upharpoonright \tau_{X^*}$ are both elements of \mathbf{M} , so is $\sigma_X^* \upharpoonright \bar{\tau}_X$. An argument similar to that in the proof of Lemma 3.8 shows that $\sigma_X^* \upharpoonright (\bar{N}_X \parallel \bar{\tau}_X)$ is fully determined by $\sigma_X^* \upharpoonright \bar{\tau}_X$, and is defined on the entire $\bar{N}_X \parallel \bar{\tau}_X$. Hence $\sigma_X^* \upharpoonright (\bar{N}_X \parallel \bar{\tau}_X) : \bar{N}_X \parallel \bar{\tau}_X \rightarrow W \upharpoonright \tau_X^* = N \upharpoonright \tau_X^*$ is a Σ_0 -preserving embedding mapping $\bar{N}_X \parallel \bar{\tau}_X$ cofinally into $N \upharpoonright \tau_X^*$. (Recall that $N \upharpoonright \tau_X^*$ is the same as $N \upharpoonright \tau_X^*$ without the top extender; and since $\bar{\tau}_X$, being a cardinal in \bar{N}_X , does not index an extender in \bar{N}_X , under a slight abuse of notation we have $\bar{N}_X \parallel \bar{\tau}_X = \bar{N}_X \upharpoonright \bar{\tau}_X$.) With a tiny bit of effort one can see that $\sigma_X^* \upharpoonright (\bar{N}_X \parallel \bar{\tau}_X)$ is fully elementary (note that $W \upharpoonright \tau_X^*$ is an elementary substructure of $W \upharpoonright \tau_{X^*}$). We thus have the following conclusion:

$$\sigma_X^* \in \mathbf{M} \text{ and } \sigma_{X^*,X} \circ \sigma_X^* \upharpoonright (\bar{N}_X \parallel \bar{\tau}_X) = \rho_X \upharpoonright (\bar{N}_X \parallel \bar{\tau}_X). \tag{13}$$

This agreement of $\sigma_{X^*,X} \circ \sigma_X^*$ with ρ_X on $\bar{N}_X \parallel \bar{\tau}_X$ makes it possible to run the argument in the proof of the Interpolation Lemma (Zeman [28, Lemma 3.6.10]), and obtain an acceptable structure N_X^* extending $N \upharpoonright \tau_X^*$ and embeddings $\tilde{\sigma}_X : \bar{N}_X \rightarrow N_X^*$ and $\sigma'_X : N_X^* \rightarrow N$ with the following properties:

- N_X^* is an end-extension of $N \upharpoonright \tau_X^*$.
- $\tau_X^* = \kappa_{X^*}^{+N_X^*}$ hence $\mathcal{P}(\kappa_{X^*}) \cap N_X^* = \mathcal{P}(\kappa_{X^*}) \cap N \upharpoonright \tau_X^* \subseteq \mathcal{P}(\kappa_{X^*}) \cap W$.
- $\tilde{\sigma}_X$ is an extension of $\sigma_X^* \upharpoonright \bar{N}_X$, and σ'_X is an extension of $\sigma_{X^*,X} \upharpoonright (N \upharpoonright \tau_X^*)$.
- Both maps $\tilde{\sigma}_X$ and σ'_X are Σ_0 -preserving, and cardinal preserving.
- $\sigma'_X \circ \tilde{\sigma}_X = \rho_X \upharpoonright \bar{N}_X$.
- N_X^*, σ'_X and $\tilde{\sigma}_X$ are elements of \mathbf{M} .

Briefly, N_X^* is constructed as the ultrapower (Zeman [28] uses the term ‘‘pseudoultrapower’’) of \bar{N}_X by σ_X^* , the map $\tilde{\sigma}_X$ is the ultrapower embedding, and σ'_X is the factor map between $\tilde{\sigma}_X$ and $\rho_X \upharpoonright \bar{N}_X$. The last item follows from the fact that the entire ultrapower construction takes place in \mathbf{M} . Since N is an initial segment of an extender model below its successor cardinal, N is a passive premouse and $N \models \text{ZFC}^-$. It follows that both maps are in fact fully elementary, and N_X^* is a passive premouse. Also, we can view σ'_X as a map from N_X^* into W , and in this case σ'_X is Σ_0 -preserving and cardinal preserving.

The above construction gives us the following for all $X, Y \in S^*$ such that $Y = Y(X)$ or $Y(X) \in Y$; note that it makes sense to form $\text{Ult}(N_X^*, F_{X^*,Y})$, as follows from the second bullet point in the paragraph below (13):

$$\begin{aligned} &\text{Either } \text{Ult}(N_X^*, F_{X^*,Y}) \text{ is ill-founded,} \\ &\text{or else } (W, \text{Ult}(N_X^*, F_{X^*,Y}), \kappa_Y) \text{ is not iterable.} \end{aligned} \tag{14}$$

The conclusion for the case where $Y(X) \in Y$ follows immediately from the conclusion for $Y = Y(X)$ since the pair (id, k) is an embedding of the phalanx $(W, \text{Ult}(W, F_{X^*,Y(X)}), \lambda_X)$ into $(W, \text{Ult}(W, F_{X^*,Y}), \kappa_Y)$ where k is the factor map between the two ultrapower embeddings. To see the conclusion for $Y = Y(X)$, recall that we put \mathcal{T}_X , an iteration tree witnessing the noniterability of the phalanx

$(N, \text{Ult}(N, F_{X,Y}), \lambda_X)$, into Z_X . Then an argument similar to the proof of Steel [21, Lemma 2.4(b)] yields that \tilde{T}_X witnesses that $(\tilde{N}_X, \tilde{N}'_X, \tilde{\lambda}_X)$ is not iterable in the sense of **M**; here recall that $\tilde{N}'_X = \text{Ult}(\tilde{N}_X, \tilde{G}_X)$ where $\tilde{G}_X = \rho_X^{-1}(G_X)$. (We stress that we are assuming $Y = Y(X)$.) Here is the place where we use the fact that θ_0 is a successor cardinal in W ; see Steel [21, Lemma 6.13] for details concerning why this assumption is useful.

Let $k' : \tilde{N}'_X \rightarrow \text{Ult}(N_{X^*}^*, F_{X^*,Y})$ be the map defined by

$$k' : [a, f]_{\tilde{G}_X} \mapsto [\rho_X(a), \tilde{\sigma}_X(f)]_{F_{X^*,Y}},$$

whenever $a \in [\tilde{\lambda}_X]^{<\omega}$ and $f \in \tilde{N}_X$ is such that $\text{dom}(f) = [\tilde{\kappa}_X]^{|\alpha|}$, where $\tilde{\kappa}_X = \text{cr}(\tilde{G}_X)$. Then the pair $(\rho_X \upharpoonright \tilde{N}_X, k')$ is an embedding of the phalanx $(\tilde{N}_X, \tilde{N}'_X, \tilde{\lambda}_X)$ into $(W, \text{Ult}(N_{X^*}^*, F_{X^*,Y}), \lambda_X)$ witnessing (14) in the case where $Y = Y(X)$. This follows by the standard computation: First notice that it follows from the definition of k' , for any $a \in [\tilde{\lambda}_X]^{<\omega}$ we have $k'(a) = k'([a, \text{id}]_{\tilde{G}_X}) = [\rho_X(a), \text{id}]_{F_{X^*,Y}} = \rho_X(a)$ and $k'(\tilde{\lambda}_X) = \lambda_X$, so we have the necessary agreement between the two maps. To see that k' is sufficiently elementary, given a formula $\varphi(v)$, Łoś theorem yields $\tilde{N}'_X \models \varphi([a, f])$ iff

$$x = \{u \in [\tilde{\kappa}_X]^{|\alpha|} \mid \tilde{N}_X \models \varphi(f(u))\} \in (\tilde{G}_X)_a.$$

By applying ρ_X , we have $\rho_X(x) \in (G_X)_{\rho_X(a)}$ or, equivalently, $\rho_X(a) \in \sigma_{X,Y}(\rho_X(x))$. Since $\sigma_{X,Y}(\rho_X(x)) = \sigma_{X,Y} \circ \sigma_X^* \circ \sigma_X^*(x) = \sigma_{X^*,Y}(\sigma_X^*(x))$, we conclude that $\sigma_X^*(x) \in (F_{X^*,Y})_{\rho(a)}$. Now

$$\sigma_X^*(x) = \{u \in [\kappa_{X^*}]^{|\alpha|} \mid N_X^* \models \varphi(\tilde{\sigma}_X(f)(u))\}$$

and, by another application of Łoś theorem, we have

$$\text{Ult}(N_X^*, F_{X^*,Y}) \models \varphi([\rho_X(a), \tilde{\sigma}_X(f)]).$$

This completes the proof of (14).

From now on, the argument closely follows Mitchell–Schimmerling [13, Section 3]. Write κ^* for κ_{X^*} . We introduce a relation $<_S$ on premece Q such that (W, Q, κ^*) is an iterable phalanx:

$Q' <_S Q$ iff there is a normal iteration tree \mathcal{T} on (W, Q, κ^*) such that Q' is an initial segment of the last model $M_\infty^\mathcal{T}$ of \mathcal{T} , and, letting b be the main branch of \mathcal{T} , one of the following holds:

- W is on b .
 - Q is on b and there is a truncation point on b .
 - Q is on b , there is no truncation point on b , and Q' is a proper initial segment of $M_\infty^\mathcal{T}$.
- (15)

The following lemma is a simple instance of Mitchell–Schimmerling [13, Lemma 3.2], rephrased for our purposes.

LEMMA 3.10. *Let Ω be the transitive closure of the singleton $\{W\}$ under the relation $<_S$. Then $<_S$ restricted to Ω is well-founded.*

PROOF. See the proof of Mitchell–Schimmerling [13, Lemma 3.2]. ⊥

The above lemma says that if we start with the phalanx (W, W, κ^*) and build a linear chain of iteration trees \mathcal{T}_i such that \mathcal{T}_0 is on (W, W, κ^*) and \mathcal{T}_{i+1} is on

(W, Q_{i+1}, κ^*) , where Q_{i+1} is the last model of \mathcal{T}_i , then for some finite n the model Q_{n+1} is on the main branch of \mathcal{T}_n and there is no truncation on the branch.

Pick $\tilde{X} \in S^*$ and let

$$S^{**} = \{Z \in S^* \mid Y(\tilde{X}) \in Z\}.$$

Note that $X^* \in Z$ whenever $Z \in S^{**}$. Lemma 3.10 is formulated for \mathbf{V} , but we apply it in \mathbf{M} for each individual $X \in S^{**}$. Given $X \in S^{**}$, either $\text{Ult}(W, F_{X^*,X})$ is ill-founded or else $(W, \text{Ult}(W, F_{X^*,X}), \kappa_X)$ is not iterable. This follows from our choice of S in (10), where the requirement is made for $G_{\tilde{X}}$ and $\lambda_{\tilde{X}}$ in place of $F_{X^*,X}$ and κ_X , respectively. However, for $X \in S^{**}$ we have $F_{X^*,X} = F_{\tilde{X},X} \circ G_{\tilde{X}} \circ F_{X^*,\tilde{X}}$ where we view extenders as maps, which, together with some little effort, which we leave to the reader, yields the above conclusion (the argument in the first two paragraphs of the proof of Lemma 3.11 can be used to get the conclusion for the composition $F_{\tilde{X},X} \circ G_{\tilde{X}}$; the presence of $F_{X^*,\tilde{X}}$ requires an additional argument, which is nevertheless easy). By Lemma 3.10 applied in \mathbf{M} , for every $X \in S^{**}$ there is a $<_S$ -minimal Q_X such that the phalanx (W, Q_X, κ^*) is iterable, and either $\text{Ult}(Q_X, F_{X^*,X})$ is ill-founded or else $(W, \text{Ult}(Q_X, F_{X^*,X}), \kappa_X)$ is not iterable. So $Q_X \in \mathbf{M}$, and the notion of iterability is also in the sense of \mathbf{M} . The next lemma says that it is possible to find a structure $X \in S^{**}$ such that Q_X can replace Q_Y for club many structures Y .

LEMMA 3.11. *There is an $X \in S^{**}$ such that for all $Y \in S^{**}$ with $X \in Y$, the following hold in \mathbf{M} :*

- (a) *Either $\text{Ult}(Q_X, F_{X^*,Y})$ is ill-founded or else $(W, \text{Ult}(Q_X, F_{X^*,Y}), \kappa_Y)$ is not iterable.*
- (b) *The premouse Q_X is $<_S$ -minimal, that is, if $Q' <_S Q_X$ then (a) fails with Q' in place of Q_X .*

PROOF. Notice that if $X \in Y$, and either $\text{Ult}(Q_X, F_{X^*,X})$ is ill-founded, or

$$(W, \text{Ult}(Q_X, F_{X^*,X}), \kappa_X)$$

is not iterable, then (a) holds, that is, either $\text{Ult}(Q_X, F_{X^*,Y})$ is ill-founded, or else the phalanx

$$(W, \text{Ult}(Q_X, F_{X^*,Y}), \kappa_Y)$$

is not iterable.

This follows from the fact that the extender $F_{X^*,Y}$ is “larger” than $F_{X^*,X}$. More precisely, let

$$k : \text{Ult}(Q_X, F_{X^*,X}) \rightarrow \text{Ult}(Q_X, F_{X^*,Y})$$

be the factor map. Then the pair (id, k) is an embedding of the phalanx $(W, \text{Ult}(Q_X, F_{X^*,X}), \kappa_X)$ into the phalanx $(W, \text{Ult}(Q_X, F_{X^*,Y}), \kappa_Y)$. Informally, since Y is “larger” than X , the structure Q_X is still “bad” in the sense that it satisfies (a), but need not be $<_S$ -minimal.

Assuming the lemma is false, for every $X \in S^{**}$ there is some $Y \in S^{**}$ with $X \in Y$, such that Q_X is not minimal such that (a) holds. Letting $X_0 = X$, construct inductively a sequence $\langle X_i \mid i \in \omega \rangle$ such that $X_i \in X_{i+1} \in S^{**}$ and $Q_{X_{i+1}} <_S Q_{X_i}$. That is, there is an iteration tree $\mathcal{T}_i \in \mathbf{M}$ on (W, Q_{X_i}, κ) such that $Q_{X_{i+1}}$ is the last model of \mathcal{T}_i , and either W is on the main branch of \mathcal{T}_i , or else Q_{X_i} is on the main branch of \mathcal{T}_i , in which case either there is a truncation on the main branch of

\mathcal{T}_i , or else $Q_{X_{i+1}}$ is a proper initial segment of the last model of \mathcal{T}_i . The sequence $\langle Q_{X_i} \mid i \in \omega \rangle$ witnesses that the relation $(<_S)^{\mathbf{M}}$ is ill-founded in the sense of \mathbf{V} . Notice that, by minimality, each Q_{X_i} is a set. By absoluteness of well-foundedness, $(<_S)^{\mathbf{M}}$ is ill-founded in the sense of \mathbf{M} , a contradiction. \dashv

Now pick X as in Lemma 3.11, and $Y \in S^{**}$ such that $X^*, X, Y(X) \in Y$. Working in \mathbf{M} , compare the phalanxes (W, Q_X, κ^*) and (W, N_X^*, κ^*) . Let \mathcal{U} and \mathcal{V} be the iteration trees arising in the comparison. That is, \mathcal{U} is on (W, Q_X, κ^*) .

Because W witnesses the soundness of $\mathbf{K}^{\mathbf{M}} \parallel \delta^+$, the standard argument shows that W cannot be on the main branches of both trees. We now get a final contradiction by ruling out all other possibilities.

We begin with the observation that Q_X must be on the main branch of \mathcal{U} . Otherwise, N_X^* is on the main branch of \mathcal{V} , as follows from the previous paragraph. Furthermore, since W is on the main branch of \mathcal{U} , and W computes successor cardinals correctly on a stationary class of cardinals, there is no truncation on the main branch of \mathcal{V} . Letting R be the last model on \mathcal{V} , we have $R <_S Q_X$, which by Lemma 3.11 means that $\text{Ult}(R, F_{X^*, Y})$ is well-founded and $(W, \text{Ult}(R, F_{X^*, Y}), \kappa_Y)$ is iterable. On the other hand, by (14) we have that either $\text{Ult}(N_X^*, F_{X^*, Y})$ is ill-founded or else $(W, \text{Ult}(N_X^*, F_{X^*, Y}), \kappa_Y)$ is not iterable. Moreover, let

$$k : \text{Ult}(N_X^*, F_{X^*, Y}) \rightarrow \text{Ult}(R, F_{X^*, Y})$$

be the map defined by $k([a, f]) = [a, \pi^{\mathcal{V}}(f) \upharpoonright [\kappa^*]^{|a|}]$, where $\pi^{\mathcal{V}} : N_X^* \rightarrow R$ is the map along the main branch of \mathcal{V} . Then the pair (id, k) is an embedding of the phalanx $(W, \text{Ult}(N_X^*, F_{X^*, Y}), \kappa_Y)$ into $(W, \text{Ult}(R, F_{X^*, Y}), \kappa_Y)$. This is a contradiction, which completes the proof that Q_X must be on the main branch of \mathcal{U} .

Let Q be the last model on \mathcal{U} , and let R , as above, be the last model on \mathcal{V} . We next argue that there is no truncation on the main branch of \mathcal{U} and $Q \trianglelefteq R$. Otherwise, a simple discussion by cases yields that there is no truncation on the main branch of \mathcal{V} , and $R \trianglelefteq Q$. Moreover, $R <_S Q_X$. Notice also that Q_X is a set, because any counterexample to iterability is witnessed by a set-sized initial segment of the model. From this it follows that N_X^* is on the main branch of \mathcal{V} . Now it is easy to see that we can get a contradiction as in the previous paragraph.

To summarize, we are left with the following situation:

- Q_X is on the main branch of \mathcal{U} .
- There is no truncation on the main branch of \mathcal{U} , and $Q \trianglelefteq R$.

Recall that the pair (id, σ'_X) is an embedding of the phalanx (W, N_X^*, κ^*) into the model W . The map σ'_X is defined below (13), and when viewed as a map from N_X^* into W , it is Σ_0 -preserving and cardinal-preserving. (Here we again use the fact that θ_0 was chosen to be a cardinal in W .) Let \mathcal{V}' be the copy of \mathcal{V} via the pair (id, σ'_X) , let R' be the last model of \mathcal{V}' , and let $\sigma' : R \rightarrow R'$ be the copy map. Also, let $\pi^{\mathcal{U}} : Q_X \rightarrow Q$ be the iteration map along the main branch of \mathcal{U} . Now $R \upharpoonright \tau_R = N_X^* \upharpoonright \tau_R$, where $\tau_R = (\kappa^*)^{+R}$ and, by the copying construction, $\sigma' \upharpoonright (R \upharpoonright \tau_R) = \sigma'_X \upharpoonright (R \upharpoonright \tau_R)$. It follows that the extender on R derived from σ'_X is compatible with $F_{X^*, X}$ and measures all sets in $\mathcal{P}(\kappa^*) \cap Q = \mathcal{P}(\kappa^*) \cap Q_X$, since there is no drop in the main branch of \mathcal{U} .

Let $\tilde{R} = \sigma'(Q)$; if $Q = R$, we of course let $\tilde{R} = R'$. Define a map $k : \text{Ult}(Q_X, F_{X^*, X}) \rightarrow \tilde{R}$ by

$$k : [a, f] \mapsto \sigma' \circ \pi^{\mathcal{U}}(f)(a)$$

for $a \in [\kappa_X]^{<\omega}$ and $f \in Q_X$ such that $\text{dom}(f) = [\kappa^*]^{|a|}$.

Using Łoś theorem, a straightforward computation similar to that in the proof of (14) shows that k is Σ_0 -preserving. It is also easy to see that k is cardinal preserving. Immediately from the definition of k we see that $k \upharpoonright \kappa_X = \text{id}$. It follows that the pair (id, k) is an embedding of the phalanx $(W, \text{Ult}(Q_X, F_{X^*.X}), \kappa_X)$ into the phalanx (W, \tilde{R}, κ_X) . Since the iteration indices in \mathcal{V} are above κ^* , the iteration indices in \mathcal{V}' are above $\sigma'_X(\kappa^*) = \kappa_X$, so (W, \tilde{R}, κ_X) is a W -generated phalanx in the sense of Steel [21, Definition 6.7]. As W is embeddable into \mathbf{K}^c , the phalanx (W, \tilde{R}, κ_X) is embeddable into a \mathbf{K}^c -generated phalanx, and is therefore iterable (see Steel [21, Lemma 6.9]). It follows that also $(W, \text{Ult}(Q_X, F_{X^*.X}), \kappa_X)$ is iterable, which contradicts the definition of Q_X .

Thus, assuming that $\text{Ult}(W, G_X)$ is ill-founded or else $(W, \text{Ult}(W, G_X), \lambda_X)$ fails to be iterable for all $X \in S$ where S comes from (10), we arrived at a final contradiction, thereby completing the proof of Theorem 1.6(a).

3.4. Proof of Theorem 1.6(b). Finally we turn to the proof of Theorem 1.6(b). Here we work under the assumption that 0^\sharp does not exist in \mathbf{M} , which means that, if we use premice with Jensen’s indexing of nonoverlapping extenders, as described in Zeman [28, Chapter 8], then the comparison process can be carried out using just linear iterations. In particular, $\mathbf{K}^{\mathbf{M}}$ is always linearly iterable in the sense of \mathbf{V} , no matter how large premice exist in \mathbf{V} . Linear iteration trees along with the condensation properties of such premice allow us, for an extender with support below δ^+ , to reduce the question of well-foundedness and iterability of $\text{Ult}(\mathbf{K}^{\mathbf{M}}, F)$ to that of well-foundedness and iterability of $\text{Ult}(\mathbf{K}^{\mathbf{M}} \parallel \delta^+, F)$, thereby avoiding any direct use of the frequent extension argument. We note here that since we deal with linear iterations, the fact that $\text{Ult}(\mathbf{K}^{\mathbf{M}}, F)$ is well-founded already implies that $\text{Ult}(\mathbf{K}^{\mathbf{M}}, F)$ is fully fine structurally iterable. We will give the details of such a reduction below. In what follows we will refer to Zeman [28, Chapter 8] on the core model theory below 0^\sharp .

So assume $\delta \geq \omega_1$ and the trees \mathcal{T}, \mathcal{U} come from the coiteration of $\mathbf{K}^{\mathbf{M}}$ against N . Thus we let $W = \mathbf{K}^{\mathbf{M}}$ in this case; otherwise we keep the rest of the notation as at the beginning of this section. There is a slight nonuniformity which comes into the argument when dealing with the cases $\delta > \omega_1$ and $\delta = \omega_1$.

If $\delta > \omega_1$ the length of the trees \mathcal{T} and \mathcal{U} is δ , as we assume, toward a contradiction, that N out-iterates $\mathbf{K}^{\mathbf{M}} \parallel \delta$. In the case where $\delta = \omega_1$, our hypothesis reads that $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$ fails to be universal with respect to countable mice in \mathbf{V} , so in this case we assume N iterates past $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$.

For the rest of the argument, write $\delta^* = \max(\delta, \omega_2)$. Notice however that even in the case where $\delta = \omega_1$, the extenders $F_{\xi, \xi'}$ are defined as in Lemma 3.6, that is, using only the trees $\mathcal{T} \upharpoonright \delta$ and $\mathcal{U} \upharpoonright \delta$. Referring back to Lemma 3.6, pick $\kappa < \lambda$ in C^* . Here $C^* \subseteq \delta$, hence $\lambda < \delta$. Recall also that, consistently with the notation fixed below (10), we write W_α for $M_\alpha^{\mathcal{T}}$ and N_α for $M_\alpha^{\mathcal{U}}$.

LEMMA 3.12. *Let $F_{\kappa, \lambda}^*$ be the extender at (κ, λ) on N_κ derived from $\pi_{\kappa, \lambda}^{\mathcal{U}}$. Then $\tilde{N} = \text{Ult}(N_{\delta^*}, F_{\kappa, \lambda}^*)$ is well-founded and iterable.*

It follows that $\text{Ult}(N_{\delta^} \parallel \delta^*, F_{\kappa, \lambda}^*)$ is well-founded and iterable.*

PROOF. By the definition of the set C^* there is an iteration map

$$\pi_{\kappa,\delta}^{\mathcal{U}} : N_{\kappa} \rightarrow N_{\delta}.$$

Write σ for $\pi_{\kappa,\delta}^{\mathcal{U}}$, and use σ to copy the linear iteration tree $\mathcal{U} \upharpoonright [\kappa, \delta^*)$ onto a linear iteration tree \mathcal{U}' on N_{δ} . This is easy to do, as the iteration trees are linear and the extenders used in these trees are internal. We thus obtain a copy map $\sigma' : N_{\delta^*} \rightarrow N'_{\delta^*}$ where we write N'_{α} for $M_{\alpha}^{\mathcal{U}'}$. Notice that if $\delta = \omega_1$, there may be a truncation on the tree $\mathcal{U} \upharpoonright [\delta, \delta^*)$, but it will not have any negative effect on the argument below. Since σ is Σ^* -preserving, so is σ' . By the choice of κ , the critical points in the tree $\mathcal{U} \upharpoonright [\kappa, \delta^*)$ are at least κ , and the iteration indices are at least $\tau'_{\kappa} = \kappa^{+N_{\delta^*}}$ (this notation is consistent with that fixed in Lemma 3.2), as $\mathcal{U} \upharpoonright [\kappa, \delta)$ does not involve any truncation. It is also easy to see that $\tau'_{\kappa} = \kappa^{+N_{\delta^*}}$. It follows that $N_{\kappa} \parallel \tau'_{\kappa} = N_{\delta^*} \parallel \tau'_{\kappa}$ and $\sigma' \upharpoonright (N_{\delta^*} \parallel \tau'_{\kappa}) = \sigma \upharpoonright (N_{\kappa} \parallel \tau'_{\kappa})$. In particular, the extender at (κ, λ) derived from σ' is precisely $F_{\kappa,\lambda}^*$. This tells us that the ultrapower $\text{Ult}(N_{\delta^*}, F_{\kappa,\lambda}^*)$ can be embedded into the premouse N'_{δ^*} via the natural factor embedding $k : [a, f]_{F_{\kappa,\lambda}^*} \mapsto \sigma'(f)(a)$ which is Σ^* -preserving, as both σ' and the ultrapower embeddings are. Now N'_{δ^*} , being an iterate of an iterable premouse, is itself iterable, which guarantees the iterability of $\text{Ult}(N_{\delta^*}, F_{\kappa,\lambda}^*)$. The last conclusion of the lemma is an immediate consequence. \dashv

We are now going to use a variant of the “shift lemma” that will allow us to embed $\text{Ult}(\mathbf{K}^{\mathbf{M}} \parallel \delta^*, F_{\kappa,\lambda})$ into $\text{Ult}(N_{\delta^*} \parallel \delta^*, F_{\kappa,\lambda}^*)$. Given structures Q, Q' , a Σ_0 -preserving map $\sigma : Q \rightarrow Q'$, extenders G at (κ, λ) on Q and G' at (κ, λ') on Q' such that $\sigma[\kappa] \subseteq \kappa$, and an order preserving map $\mathfrak{k} : \lambda \rightarrow \lambda'$, we write $(\sigma, \mathfrak{k}) : (Q, G) \rightarrow (Q', G')$ if and only if for every $a \in [\lambda]^{<\omega}$ and $x \in \mathcal{P}([\kappa]^{|a|}) \cap Q$,

$$x \in G_a \implies \sigma(x) \cap [\kappa]^{|a|} \in G'_{\mathfrak{k}[a]}.$$

This corresponds to the similar notion discussed in Zeman [28, Section 2.5]. The slight difference between the notion used here and that in Zeman [28] is that, in our situation, it may happen that $\sigma(\text{cr}(G)) > \text{cr}(G')$. If $(\sigma, \mathfrak{k}) : (Q, G) \rightarrow (Q', G')$, then we can run the proof of the shift lemma and show that there is precisely one Σ_0 -preserving embedding

$$\sigma' : \text{Ult}(Q, G) \rightarrow \text{Ult}(Q', G')$$

such that $\sigma' \upharpoonright \lambda = \mathfrak{k}$ and $\sigma' \circ \pi_G = \pi_{G'} \circ \sigma$, where $\pi_G, \pi_{G'}$ are the corresponding ultrapower embeddings. Moreover, if $Q \models \text{ZFC}^-$ and σ is fully elementary, so is σ' .

LEMMA 3.13. *Let $\kappa < \lambda$ be in C^* , and let $F_{\kappa,\lambda}^*$ be as in Lemma 3.12. Then the following hold:*

- (a) $(\pi_{0,\delta^*}^{\mathcal{T}} \upharpoonright (\mathbf{K}^{\mathbf{M}} \parallel \delta^*), \pi_{0,\delta^*}^{\mathcal{T}} \upharpoonright \lambda) : (\mathbf{K}^{\mathbf{M}} \parallel \delta^*, F_{\kappa,\lambda}) \rightarrow (N_{\delta^*} \parallel \delta^*, F_{\kappa,\lambda}^*)$.
- (b) $\text{Ult}(\mathbf{K}^{\mathbf{M}} \parallel \delta^*, F_{\kappa,\lambda})$ is well-founded and iterable.

PROOF. Clause (a) follows by a straightforward computation using the fact that $\text{cr}(\pi_{\lambda,\delta^*}^{\mathcal{T}}) \geq \lambda$. Recall that we are using Jensen’s indexing of nonoverlapping extenders. This computation uses elementary facts about this indexing. Clause (b) follows from (a) and Lemma 3.12. \dashv

Now let $S = \mathcal{P}_{\delta}(\delta^+) \cap \mathbf{M}$, pick θ large regular in \mathbf{V} , and let S^θ be as in Lemma 3.8. If we pick $X \in Y$ such that both $X, Y \in S^\theta$, and let $\kappa = \kappa_X$ and $\lambda = \kappa_Y$, then $F_{\kappa,\lambda} \in \mathbf{M}$. From now on, we work entirely in \mathbf{M} . By Lemma 3.13, $\text{Ult}(\mathbf{K}^{\mathbf{M}} \parallel \delta^*, F_{\kappa,\lambda})$ is well-founded and iterable. We show that this conclusion can be extended to $\mathbf{K}^{\mathbf{M}}$.

LEMMA 3.14. $\text{Ult}(\mathbf{K}^{\mathbf{M}}, F_{\kappa,\lambda})$ is well-founded and iterable.

PROOF. Assume the contrary. Working in \mathbf{M} , let ν be a $\mathbf{K}^{\mathbf{M}}$ -cardinal large enough such that, letting $Q = \mathbf{K}^{\mathbf{M}} \parallel \nu$, the ultrapower $\text{Ult}(Q, F_{\kappa,\lambda})$ is either ill-founded or not iterable, and let \mathcal{V} be a linear iteration tree on Q in \mathbf{M} witnessing this. Again, if $\text{Ult}(Q, F_{\kappa,\lambda})$ is ill-founded, we let \mathcal{V} be the tree of length 1 consisting of this ill-founded ultrapower, in order to treat the two cases uniformly. Let θ^* be a large regular cardinal such that $\mathcal{V} \in H_{\theta^*}^{\mathbf{M}}$, and let Z be an elementary substructure of $H_{\theta^*}^{\mathbf{M}}$ of size less than δ such that $F_{\kappa,\lambda}, \mathcal{V} \in Z$ and $\bar{\delta} = Z \cap \delta \in \delta$. Also, let H be the transitive collapse of Z , and let $\sigma : H \rightarrow H_{\theta^*}^{\mathbf{M}}$ be the inverse to the Mostowski collapsing isomorphism. Notice that $\bar{\delta} > \lambda$ and $\sigma(F_{\kappa,\lambda}) = F_{\kappa,\lambda}$. Letting $(\bar{Q}, \bar{\mathcal{V}}) = \sigma^{-1}(Q, \mathcal{V})$, the linear iteration tree $\bar{\mathcal{V}}$ witnesses the ill-foundedness/noniterability of $\text{Ult}(\bar{Q}, F_{\kappa,\lambda})$. One can of course focus merely on the proof of well-foundedness, as for linear iterability, well-foundedness implies iterability (as already mentioned above). However, the proof is the same for both, and does not need to rely on this fact. We will use the following observation which, as was pointed out to us by the referee, can also be proved for premice with Mitchell–Steel indexing:

CLAIM 3.15. Let $\mathcal{R}, \bar{\mathcal{R}}$ be the linear iteration trees on Q, \bar{Q} , respectively, coming from the comparison of Q against \bar{Q} . Then the critical points in $\bar{\mathcal{R}}$ are above $\bar{\delta}$.

PROOF. If $\bar{\delta}$ is not overlapped by an extender in \bar{Q} then this is immediate. So suppose $\bar{\delta}$ is overlapped. Here we apply a condensation result particular for the indexing of the extenders we are using here; recall this is the indexing described in Zeman [28, Chapter 8]. It follows that either \bar{Q} is a proper initial segment of Q or else there is $\mu < \bar{\delta}$ and $\beta > \bar{\delta}$ such that $E_{\beta}^{\bar{Q}} = \emptyset$, $\text{cr}(E_{\alpha}^{\bar{Q}}) = \mu$ whenever $\mu^{+\bar{Q}} \leq \alpha < \beta$ and α indexes an extender in \bar{Q} , and for every $\alpha > \beta$ that indexes an extender in \bar{Q} we have $\text{cr}(E_{\alpha}^{\bar{Q}}) > \beta$. In the former case the comparison of Q against \bar{Q} is trivial, so let us focus on the latter case.

Since $\sigma : \bar{Q} \rightarrow Q$ is elementary and $\text{cr}(\sigma) = \bar{\delta}$ then, by Zeman [28, Lemma 8.2.2], $E_{\beta}^{\bar{Q}} \upharpoonright \beta = E_{\beta}^Q \upharpoonright \beta$. It follows that in the coiteration of Q against \bar{Q} , all iteration indices used in $\bar{\mathcal{R}}$ are larger than β , but by the discussion above, also the corresponding critical points are above β , and hence above $\bar{\delta}$. \dashv

We can now complete the proof of the lemma. Since $\delta^* \geq \omega_2^{\mathbf{M}}$, the initial segment $\mathbf{K}^{\mathbf{M}} \parallel \delta^*$ is universal for all premice in \mathbf{M} of size less than δ^* ; this follows by adjusting the proof for Steel’s core model in Schimmerling–Steel [18] to the indexing we are currently using. Hence $\mathbf{K}^{\mathbf{M}} \parallel \delta^*$ iterates past \bar{Q} , and the coiteration terminates after less than δ^* steps. Let \mathcal{R}' be the linear iteration tree on the $\mathbf{K}^{\mathbf{M}} \parallel \delta^*$ -side of the coiteration of \bar{Q} against $\mathbf{K}^{\mathbf{M}} \parallel \delta^*$. Since $\mathbf{K}^{\mathbf{M}} \parallel \delta^* \trianglelefteq Q$ and δ^* is a regular cardinal, the iteration tree on the \bar{Q} -side of this coiteration is precisely $\bar{\mathcal{R}}$, and the iteration tree \mathcal{R}' is essentially the same as \mathcal{R} with the only difference that $M_{\alpha}^{\mathcal{R}'} = M_{\alpha}^{\mathcal{R}} \parallel \delta^*$ for all $\alpha < \text{lh}(\mathcal{R})$. By the above discussion, there is an iteration map $\pi_{\infty}^{\bar{\mathcal{R}}} : \bar{Q} \rightarrow M_{\infty}^{\bar{\mathcal{R}}}$ where $M_{\infty}^{\bar{\mathcal{R}}}$ is the last model on $\bar{\mathcal{R}}$. Moreover, $M_{\infty}^{\bar{\mathcal{R}}} \trianglelefteq M_{\infty}^{\mathcal{R}'}$. By the Claim, $\text{cr}(\pi_{\infty}^{\bar{\mathcal{R}}}) \geq \bar{\delta} > \lambda$. It follows that $\text{Ult}(\bar{Q}, F_{\kappa,\lambda})$ can be embedded into $\text{Ult}(M_{\infty}^{\bar{\mathcal{R}}}, F_{\kappa,\lambda})$ by $[a, f]_{F_{\kappa,\lambda}} \mapsto [a, \pi_{\infty}^{\bar{\mathcal{R}}}(f)]$, which shows that $\text{Ult}(M_{\infty}^{\bar{\mathcal{R}}}, F_{\kappa,\lambda})$ is either ill-founded, or else not iterable. It follows that $\text{Ult}(M_{\infty}^{\mathcal{R}'}, F_{\kappa,\lambda})$ is ill-founded or not iterable.

On the other hand, $Q^* = \text{Ult}(\mathbf{K}^M \parallel \delta^*, F_{\kappa,\lambda})$ is well-founded and iterable by Lemma 3.13(b). Let ρ be the associated ultrapower map. Use ρ to copy the linear iteration tree \mathcal{R}' onto a linear iteration tree \mathcal{R}^* on Q^* via the map ρ . Let $\rho^* : M_{\infty}^{\mathcal{R}'} \rightarrow M_{\infty}^{\mathcal{R}^*}$ be the copy map between the last two models in the copy construction. Since the iteration indices of \mathcal{R}' are above $\bar{\delta}$, the map ρ^* agrees with ρ below $\bar{\delta}$. It follows that the extender on $M_{\infty}^{\mathcal{R}'}$ at (κ, λ) derived from ρ^* is precisely $F_{\kappa,\lambda}$, hence $\text{Ult}(M_{\infty}^{\mathcal{R}'}, F_{\kappa,\lambda})$ can be embedded into $M_{\infty}^{\mathcal{R}^*}$ via the natural map $[a, f]_{F_{\kappa,\lambda}} \mapsto \rho^*(f)(a)$. As \mathcal{R}_{∞}^* , being an iterate of an iterable premouse, is itself iterable, this proves the iterability of $\text{Ult}(M_{\infty}^{\mathcal{R}'}, F_{\kappa,\lambda})$. Now we have reached a contradiction with the conclusion of the previous paragraph, thereby completing the proof of Lemma 3.14. \dashv

We can now finish the proof of Theorem 1.6(b). Here we could use a criterion on absorption of extenders by \mathbf{K} which can be found on page 274 in Chapter 8 in Zeman [28] and which, as pointed out by the referee, needs some corrections. However, to keep the text more uniform, we use a variant of Lemma 2.4 for the current indexing. Let $\mathbf{K}' = \text{Ult}(\mathbf{K}^M, F_{\kappa,\lambda})$. Then \mathbf{K}^M coiterates with \mathbf{K}' above λ , as otherwise \mathbf{K} would have an extender overlapping λ which would imply that also N_{δ} has such an extender; see the paragraph immediately above Lemma 3.12 for notation. Our indexing however does not allow this, as this would mean that in at least one of the preimage N_i there is an overlap of extenders. We now coiterate \mathbf{K}^M against \mathbf{K}' and, as the critical points of the coiteration are above λ , similarly as in the proof of Lemma 2.4 we conclude that $F_{\kappa,\lambda} \in \mathbf{K}^M$. As in that proof, this gives a contradiction since $F_{\kappa,\lambda}$ codes a map which collapses $\lambda^{+\mathbf{K}^M}$ in \mathbf{K}^M . This completes the proof of Theorem 1.6(b).

3.5. An improvement suggested by the referee. The following improvement of Theorem 1.6, which we record as Theorem 3.16, was suggested by the referee. We first introduce some terminology, also suggested by the referee.

Let δ be regular and P be a premouse of height δ . We say that P is neatly universal with respect to a class of preimage \mathcal{S} iff for every $N \in \mathcal{S}$ there is a terminal comparison $(\mathcal{T}, \mathcal{U})$ of P against N such that $\mathcal{M}_{\infty}^{\mathcal{U}}$ is an initial segment of $\mathcal{M}_{\infty}^{\mathcal{T}}$, there is no drop on the main branch of \mathcal{U} , the comparison uses only extenders of length $< \delta$, and if the comparison is of length δ then $i^{\mathcal{U}}[\delta] \subseteq \delta$ where $i^{\mathcal{U}}$ is the iteration map along the main branch of \mathcal{U} .

We say that a premouse N is short iff N is 1-small and, for some $n \in \omega$, N is n -sound and for every n -maximal normal Q -structure guided iteration tree \mathcal{T} on N of limit length, the Q -structure for \mathcal{T} exists and is of the form $J_{\alpha}(\mathcal{M}(\mathcal{T}))$ for some ordinal α . We say that a short premouse is short tree normally iterable iff N is normally iterable with respect to the Q -structure guided strategy.

Let δ be regular. A premouse R of height δ is anomalous iff R has a largest cardinal μ which is singular in R and, letting $\kappa = \text{cf}^R(\mu)$, there are unboundedly many $\alpha < \kappa^{+R}$ such that $\text{cr}(E_{\alpha}^R) = \kappa$. The point of introducing the notion of anomalous premouse is that if P is an anomalous premouse which has a total extender on its sequence with critical point $\text{cf}^P(\mu_P)$ where μ_P witnesses that P is anomalous then the ultrapower by such an extender has height strictly larger than δ . This may cause potential problems with standard arguments when attempting a proof of universality of R .

THEOREM 3.16. *Let $\delta > \omega$ be regular, and let \mathbf{M} be a proper class inner model of ZFC. Then the following hold.*

- (a) *Assume that there is no proper class inner model with a Woodin cardinal from the point of view of \mathbf{M} , and S_δ is stationary in the sense of \mathbf{V} . Then $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is neatly universal with respect to all $(\delta + 1)$ -short tree normally iterable short premeice of height $< \delta$ in \mathbf{V} . Moreover, if $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is not anomalous, then “ $< \delta$ ” can be replaced with “ $\leq \delta$ ”.*
- (b) *Assume \mathbf{M} believes that 0^\sharp does not exist, and $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary. If $\delta > \omega_1^{\mathbf{M}}$ then $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is neatly universal with respect to all $(\delta + 1)$ -iterable premeice of height $< \delta$, and if $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is not anomalous then “ $< \delta$ ” can be replaced with “ $\leq \delta$ ”. If $\delta = \omega_1^{\mathbf{M}}$ (hence $= \omega_1^{\mathbf{V}}$) then $\mathbf{K}^{\mathbf{M}} \parallel \omega_2^{\mathbf{M}}$ is universal with respect to all countable premeice, and if $\mathbf{K}^{\mathbf{M}} \parallel \omega_2^{\mathbf{M}}$ is not anomalous then “countable” can be replaced with “of height $\leq \omega_1$ ”.*

In the case of (a), our argument can be run as the comparison of N against $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is successful for premeice as in (a) under the weaker hypothesis that $\mathbf{M} \models$ “There is no proper class inner model with a Woodin cardinal”, and a closer inspection of our frequent extension argument reveals that it merely used $\delta > \omega_1^{\mathbf{M}}$ rather than $\delta > \omega_1$.

In the case of (b), if $\delta > \omega_1$ all we really used in the proof of Theorem 1.6(b) was that $\delta^* > \omega_1^{\mathbf{M}}$. (See the paragraph immediately preceding Lemma 3.12 for the definition of δ^* .) If $\delta = \omega_1^{\mathbf{M}}$ we similarly only used that $\delta^* \geq \omega_2^{\mathbf{M}}$. There was another place in the proof where we referred to ω_2 , namely when we argued that N iterates past $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$, but one can easily check that the argument goes through if we merely know that N iterates past $\mathbf{K}^{\mathbf{M}} \parallel \omega_2^{\mathbf{M}}$.

§4. Discussion. In this section we discuss some issues concerning why we are unable to prove our main result in its full generality, and outline a possible approach that may lead to removing all our restricting assumptions, at least in the absence of Woodin cardinals. Our conjecture thus can be formulated as follows:

CONJECTURE 4.1. *Assume that there is no proper class inner model with a Woodin cardinal. Let δ be an uncountable regular cardinal, and let \mathbf{M} be a proper class inner model such that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary. Then the following hold:*

- (a) *If $\delta > \omega_1^{\mathbf{M}}$, then $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is universal for all iterable premeice in \mathbf{V} of cardinality less than δ .*
- (b) *If $\delta = \omega_1^{\mathbf{M}}$ (hence $\omega_1^{\mathbf{M}} = \omega_1^{\mathbf{V}}$), then $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$ is universal for all countable iterable premeice in \mathbf{V} .*

Theorem 1.6(b) says that the above conclusions (a) and (b) hold under the rather restrictive anti-large cardinal hypothesis that a sharp for an inner model with a strong cardinal does not exist. Unfortunately, we do not know how to extend our proof of Theorem 1.6(b) to the more general situation with no inner models with Woodin cardinals. We do, however, have some partial results along these lines that we proceed to describe.

4.1. The case $\delta > \omega_1^{\mathbf{M}}$. One drawback of our proof of Theorem 1.6(a) is the direct use of the frequent extension argument. As we have already explained above, the use of this argument requires us to work with the set S_δ from (2) in place of $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$, as it relies on the fact that countable subsets of structures in S_δ can be

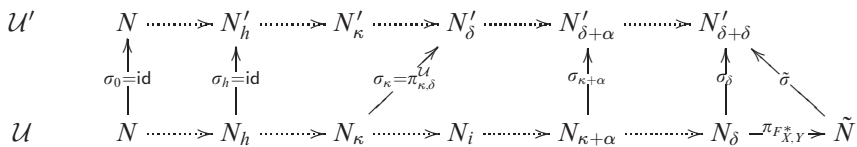
internally covered. And, as explained before, there is no way around, by the results in Räsch–Schindler [17] and references therein. Our proof of Theorem 1.6(b) avoids the direct use of the frequent extension argument by showing that for suitable δ and X, Y , (i) $\text{Ult}(\mathbf{K}^{\mathbf{M}} \parallel \delta, F_{X,Y})$ is well-founded and iterable, and (ii) the question of well-foundedness and iterability of $\text{Ult}(\mathbf{K}^{\mathbf{M}}, F_{X,Y})$ can be reduced to that of the well-foundedness and iterability of $\text{Ult}(\mathbf{K}^{\mathbf{M}} \parallel \delta, F_{X,Y})$. Here, (ii) relies on the result from Schimmerling–Steel [18] that $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is universal for iterable premice of size less than δ in \mathbf{M} whenever $\delta \geq \omega_2^{\mathbf{M}}$. Under the assumption “there is no proper class inner model with a Woodin cardinal”, (ii) needs to be replaced with a stronger statement, namely that $\text{Ult}(W, F_{X,Y})$ is well-founded and the phalanx $(W, \text{Ult}(W, F_{X,Y}), \kappa_Y)$ is iterable, where W is the extender model witnessing the soundness of a suitably long initial segment of $\mathbf{K}^{\mathbf{M}}$.

In the following, we outline a way of generalizing the approach from the proof of Theorem 1.6(b) for $\delta > \omega_1$ to the situation where we assume there are no proper class inner models with a Woodin cardinal. The argument from the proof of Theorem 1.6(b) can be generalized to obtain the following:

$$\begin{aligned} & \text{Ult}(\mathbf{K}^{\mathbf{M}} \parallel \delta, F_{X,Y}) \text{ is well-founded, and} \\ & (\mathbf{K}^{\mathbf{M}} \parallel \delta, \text{Ult}(\mathbf{K}^{\mathbf{M}} \parallel \delta, F_{X,Y}), \kappa_Y) \text{ is iterable.} \end{aligned} \tag{16}$$

However, we do not know how to use (16) along with the universality of $\mathbf{K}^{\mathbf{M}} \parallel \delta$ to prove the iterability of the phalanx $(W, \text{Ult}(W, F_{X,Y}), \kappa_Y)$.

The conclusion (16) is established using a copying construction that generalizes that in the proof of Theorem 1.6(b). More precisely, letting $\kappa = \kappa_X$, if \mathcal{U} is the normal iteration tree on N coming from the coiteration with W as described below (4), we can “embed” $\mathcal{U} \upharpoonright \kappa$ into \mathcal{U} using a system of maps σ_i where $\sigma_\alpha : M_\alpha^{\mathcal{U}} \rightarrow M_\alpha^{\mathcal{U}}$ is the identity map for $\alpha < \kappa$, and $\sigma_\kappa : M_\kappa^{\mathcal{U}} \rightarrow M_\delta^{\mathcal{U}}$ is the iteration map $\pi_{\kappa,\delta}^{\mathcal{U}}$. A straightforward, but a bit tedious computation shows that we can copy the iteration tree \mathcal{U} onto a normal iteration tree \mathcal{U}' of length $\delta + \delta$ extending \mathcal{U} , where the copying construction uses the maps σ_i instead of a single map. The point here is to verify that there are no “conflicts” when we copy, that is, \mathcal{U}' is indeed a normal iteration tree. The situation can be then depicted by the following diagram; we write N_α for $M_\alpha^{\mathcal{U}}$ and N'_α for $M_\alpha^{\mathcal{U}'}$.



As the iteration indices in $\mathcal{U} \upharpoonright [\kappa, \delta)$ are larger than κ , the copy map σ_δ agrees with σ_κ on $\mathcal{P}(\kappa) \cap N_\kappa = \mathcal{P}(\kappa) \cap N_\delta$. It follows that the two derived extenders agree, and in fact they agree with the map $\sigma_X \upharpoonright (\mathcal{P}(\kappa) \cap N_\delta)$. (See the settings above (7)). So if we pick Y such that $X \in Y$, and let $\lambda = \kappa_Y$ and $F_{X,Y}^*$ be the (κ, λ) -extender derived from $\sigma_{X,Y} \upharpoonright (\mathcal{P}(\kappa) \cap N_\delta)$, we see that $\tilde{N} = \text{Ult}(N_\delta, F_{X,Y}^*)$ is well-founded, and the factor map $\tilde{\sigma} : \tilde{N} \rightarrow N'_{\delta+\delta}$ is the identity below λ , and sends λ to δ .

It follows that the pair $(\text{id}, \tilde{\sigma})$ is an embedding of the phalanx $(N'_{\delta+\delta}, \tilde{N}, \lambda)$ into $N'_{\delta+\delta}$, which proves the iterability of the phalanx. This of course yields the iterability of the phalanx $(N'_{\delta+\delta} \parallel \delta, \tilde{N} \parallel \delta, \lambda)$. Note that $N'_{\delta+\delta} \parallel \delta = N_\delta \parallel \delta$. Recall

that \mathcal{T} is the iteration tree on the W -side of the coiteration of N with W ; see again the settings below (4). As in the proof of Theorem 1.6(b), we can then show that the phalanx $(\mathbf{K}^M \parallel \delta, \text{Ult}(\mathbf{K}^M \parallel \delta, F_{X,Y}), \lambda)$ can be elementarily embedded into $(N'_{\delta+\delta} \parallel \delta, \tilde{N} \parallel \delta, \lambda)$ using the pair of maps $(\pi_{0,\delta}^{\mathcal{T}} \upharpoonright (\mathbf{K}^M \parallel \delta), k)$ where $k : \text{Ult}(\mathbf{K}^M \parallel \delta, F_{X,Y}) \rightarrow \tilde{N} \parallel \delta$ is the unique elementary map that agrees with $\pi_{0,\delta}^{\mathcal{T}}$ below λ and satisfies $k \circ \pi_{X,Y} = \pi_{X,Y}^* \circ \pi_{0,\delta}^{\mathcal{T}}$, and $\pi_{X,Y}^*$ and $\pi_{X,Y}$ are the ultrapower embeddings coming from $\text{Ult}(N_\delta \parallel \delta, F_{X,Y}^*)$ and $\text{Ult}(\mathbf{K}^M \parallel \delta, F_{X,Y})$, respectively. This completes the sketch of the proof of (16).

4.2. The case $\delta = \omega_1^M = \omega_1^V$. Although superficially this case may appear to be very special, it is interesting in many respects. The main difference between this case and the one discussed in Subsection 4.1 is that, as we already mentioned in the introduction, we cannot expect $\mathbf{K}^M \parallel \delta$ to be universal for countable mice, by the result of Jensen [10]. Suppose we try to prove Conjecture 4.1 using the strategy from Subsection 4.1. By the fact that $\mathbf{K}^M \parallel \delta$ is no longer universal for countable premice, establishing (16) is not sufficient to run the argument. It seems that what we would need here is the variant of (16) where δ is replaced with δ^+ , and this in turn requires to use the coiteration of W against N which has length δ^+ . However, the extender $F_{X,Y}$ is determined already by the coiteration of length δ . If we are in the situation where no sharp for an inner model with a strong cardinal exists, all iteration trees can be made linear, so once the well-foundedness of $\text{Ult}(N_{\delta^+}, F_{X,Y}^*)$ has been established, one can easily embed $\text{Ult}(\mathbf{K}^M \parallel \delta^+, F_{X,Y})$ into $\text{Ult}(N_{\delta^+}, F_{X,Y}^*)$ since $\pi_{0,\delta^+}^{\mathcal{T}} = \pi_{\delta,\delta^+}^{\mathcal{T}} \circ \pi_{0,\delta}^{\mathcal{T}}$ and the critical point of $\pi_{\delta,\delta^+}^{\mathcal{T}}$ is at least δ . In the situation with many strong cardinals, we have nonlinear iteration trees, so even if we succeeded in running an argument similar to that outlined in Subsection 4.1 and proved that $(N_{\delta^+}, \text{Ult}(N_{\delta^+}, F_{X,Y}^*), \kappa_Y)$ is iterable, the argument for linear iterations for “pulling” back to \mathbf{K}^M we have just mentioned can be mimicked here only if we know (among other things) that δ is on the branch $[0, \delta^+]_{\mathcal{T}}$, or, more generally, if some sufficiently large element of the set C^* defined below Lemma 3.1 is on that branch. It is also easy to see that if δ is not on the branch $[0, \delta^+]_{\mathcal{T}}$ then there is a sharp for an inner model with a strong cardinal, but we do not know if a substantially stronger large cardinal hypothesis can be extracted in this situation. (For instance an inner model with two strong cardinals.)

4.3. No inner models of M with Woodin cardinals. Finally, it is natural to wonder whether the anti-large cardinal assumption in Theorem 1.6(a) that there are no inner models with Woodin cardinals can be localized to M .

In this setting, we do not know if proper class extender models of M are iterable in V . We can however prove a weaker form of iterability for such models, and establish a weak form of our main theorem. The problem is that this weak form does not seem to suffice to deduce any interesting applications, so we omit its proof. The precise statement of the result we have in this case is as follows:

THEOREM 4.2. *Assume that M is a proper class inner model, and $\delta > \omega_1$ is a regular cardinal in V such that S_δ is stationary. Granting that in M there is no proper class inner model with a Woodin cardinal, the initial segment $\mathbf{K}^M \parallel \delta$ is universal for all iterable 1-small premice in V of cardinality less than δ in the following weak sense:*

If N is a 1-small iterable premouse in V of cardinality less than δ , then there is a pair of iteration trees $(\mathcal{T}, \mathcal{U})$ of length less than δ , where \mathcal{T} is on $\mathbf{K}^M \parallel \delta$ and \mathcal{U} is on N ,

such that \mathcal{U} has a last model $M_\infty^{\mathcal{U}}$, there is no truncation on the main branch of \mathcal{U} , and one of the following holds:

- (a) The tree \mathcal{T} has a last well-founded model $M_\infty^{\mathcal{T}}$, and $M_\infty^{\mathcal{U}}$ is an initial segment of $M_\infty^{\mathcal{T}}$.
- (b) The tree \mathcal{T} has a limit length, does not have a cofinal well-founded branch, and $\delta(\mathcal{T}) = \delta(\mathcal{U})$. Moreover, no ordinal larger than $\delta(\mathcal{U})$ indexes an extender on $M_\infty^{\mathcal{U}}$ and, in some generic extension of \mathbb{V} , there is a cofinal branch b through \mathcal{T} such that the direct limit along b has well-founded part of length at least $\text{On} \cap M_\infty^{\mathcal{U}}$.

The interest here, of course, would be to replace this weak universality with its genuine version or, if this is not possible, to see whether this version can actually be useful. For example, this version seems irrelevant for Σ_3^1 -correctness, for which one needs to assume that there are no inner models with a Woodin cardinal in \mathbb{V} .

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