The Journal of Symbolic Logic
Volume 76, Number 4, Dec. 2011

BPFA AND PROJECTIVE WELL-ORDERINGS OF THE REALS

ANDRÉS EDUARDO CAICEDO AND SY-DAVID FRIEDMAN

Abstract. If the bounded proper forcing axiom BPFA holds and $\omega_1 = \omega_1^L$, then there is a lightface Σ^1_3 well-ordering of the reals. The argument combines a well-ordering due to Caicedo-Veličković with an absoluteness result for models of MA in the spirit of "David's trick." We also present a general coding scheme that allows us to show that BPFA is equiconsistent with R being lightface Σ^1_4, for many "consistently locally certified" relations R on $\mathbb{R}$. This is accomplished through a use of David's trick and a coding through the Σ_2 stable ordinals of L.

§1. Introduction. Throughout this paper, *forcing* means set forcing. BPFA denotes the Bounded Proper Forcing Axiom introduced in Goldstern-Shelah [9]. Recall that BPFA is equivalent to the assertion that $H(\omega_2) \prec_{\Sigma_1} V^{\mathbb{P}}$ for any proper forcing $\mathbb{P}$, see Bagaria [1].

In this paper we show that BPFA implies the existence of well-orderings of descriptive set theoretic optimal complexity under the anti-large cardinal assumption that $\omega_1 = \omega_1^L$.

Recall that $\vec{C} = (C_\alpha : \alpha < \omega_1)$ is a *C-sequence* (or a *ladder system*) iff $C_\alpha \subseteq \alpha$ is cofinal in α and of least possible order type, for all $\alpha < \omega_1$.

In Caicedo-Veličković [5] it is shown that BPFA implies that for any C-sequence $\vec{C}$ there is a Δ_1 well-ordering of $\mathbb{R}$ in $\vec{C}$ as a parameter. The proof requires an understanding of the theory of the Mapping Reflection Principle MRP, see Moore [12].

Here, we combine this result with a coding method of David (see Friedman [6] or [7, §6.2]) to prove:

Theorem 1. *If* BPFA *holds and* $\omega_1 = \omega_1^{L[r]}$ *for some real* r, *then there is a* $\Sigma^1_3(r)$ *well-ordering of the reals.*

Notice that we obtain an implication rather than merely a consistency result. The conclusion is best possible in the sense that already MA_{ω_1} (Martin's axiom for partial orders of size ω_1) implies that $\boldsymbol{\Sigma}^1_2$ sets are Lebesgue measurable, and therefore there are no $\boldsymbol{\Sigma}^1_2$ well-orderings; this goes back to Martin-Solovay [11].

Something like the smallness assumption that some $L[r]$ computes ω_1 correctly is needed in Theorem 1. For example: Assuming that every real has a sharp, the existence of a $\boldsymbol{\Sigma}^1_3$ well-ordering of the reals implies CH. In addition, in the presence of sharps, MA_{ω_1} implies that every $\boldsymbol{\Sigma}^1_3$ set of reals is Lebesgue measurable. These two statements are proved in Hjorth [10].

Received November 25, 2009.

1943-5886/11/7604-0002/$2.10

Recall that a cardinal κ is *reflecting* iff κ is regular and V_κ is Σ_2-elementary in the universe V. Reflecting cardinals relativize down to inner models of the form $L[r]$ for any real r. In Goldstern-Shelah [9] it is shown that BPFA can be forced in the presence of a reflecting cardinal; in fact, we have an equiconsistency, since BPFA implies that ω_2 is reflecting in $L[r]$ for any real r.

Combining these observations with Theorem 1, we immediately obtain:

COROLLARY 2. *The following are equiconsistent*:

1. *There is a proper class of reflecting cardinals.*
2. *Any forcing extension of V admits a further forcing extension where* BPFA *holds and there is a $\utilde{\Sigma}^1_3$ well-ordering of the reals.*

PROOF. (2) implies that there is a proper class of cardinals that are the ω_2 of some forcing extension where BPFA holds. All these cardinals are reflecting in L, giving (1).

Conversely, (1) implies that there is a proper class of reflecting cardinals in L. It follows that L satisfies (2), because reflecting cardinals are preserved by small forcing, and no forcing extension of L has any sharps. But if V is not closed under sharps, it is easy to pass to a forcing extension W where there is some real r such that $\omega_1^W = \omega_1^{L[r]}$, see for example, Caicedo-Schindler [4]. (Much stronger *reshaping* results are possible in this situation; this was first noticed in Shelah-Stanley [16] and is implicit in earlier work by Jensen.) But then forcing over W with the standard poset for BPFA gives us a model where Theorem 1 applies. ⊣

For the definition of *remarkable cardinals* see Schindler [14]. From the results there, it follows that if ω_1 is not remarkable in L then there is a proper forcing $\mathbb{P}$ adding a real r such that $\omega_1 = \omega_1^{L[r]}$. We therefore have:

COROLLARY 3. *Assume that ω_1 is not remarkable in L. Then* BPFA *implies that there is a $\utilde{\Sigma}^1_3$ well-ordering of the reals.* ⊣

It was shown in Caicedo [3] that BPFA is consistent with the existence of projective well-orderings of the reals, and it was already noted in Caicedo-Veličković [5] that if $\omega_1^L = \omega_1$ and BPFA holds, then there is a *lightface* projective well-ordering. However, the coding arguments used in these papers do not seem to suffice to obtain a well-ordering of smaller complexity than Σ^1_6.

It is shown in Friedman [7, Theorem 8.51] that MA $+ \omega_1 = \omega_1^L$ is consistent with a Σ^1_3 well-ordering. The argument uses an iteration of almost disjoint codings. A natural attempt by the second author at generalizing this approach using Jensen-like codings failed because we do not have the kind of reflection needed to ensure BPFA at the end of the iteration—while the kind of reflection required by MA poses no difficulties.

Originally we obtained a general coding argument that in particular gives us the weaker result that from optimal hypotheses, BPFA is consistent with a Σ^1_4 well-ordering of the reals. The well-ordering of optimal complexity is obtained by combining some of the ideas in that argument with the result from Caicedo-Veličković [5]. What we actually prove is the the following:

THEOREM 4. *Assume that MA_{ω_1} holds and $\omega_1 = \omega_1^{L[r]}$ for some real r. Let $R(\vec{x})$ be a Σ_1 relation on reals with ω_1 as a parameter. Then R is $\Sigma^1_3(r)$.*

Theorem 1 follows immediately from Theorem 4. This is proved in Section 2. Since our original coding argument contains ideas that the reader may find of independent interest, we present it in Section 3. The paper closes with some questions in Section 4.

1.1. Acknowledgements. We want to thank the editors for their patience and the anonymous referee, whose comments greatly improved the presentation of the paper. The first author thanks the National Science Foundation for partial support through grant DMS-0801189. The second author wishes to thank the Austrian Science Fund (FWF) for its generous support through Project Number P 19375-N18.

§2. Σ_1-in-ω_1 statements are Σ^1_3.

In this section we prove Theorems 1 and 4. Let us fix some notation. Let ZFC^- denote ZFC without the power set axiom. For $B \subseteq \omega_1$, let

$$\mathrm{Even}(B) = \{\delta \mid 2\delta \in B\}$$

and

$$\mathrm{Odd}(B) = \{\delta \mid 2\delta + 1 \in B\}.$$

To avoid carrying an additional parameter around, we assume from now on that $\omega_1 = \omega_1^L$.

We begin with Theorem 4. Assume MA_{ω_1}.

Let $R(\vec{x}, z)$ be a Σ_1 formula such that $R(\vec{x}, \omega_1)$ defines a relation on reals. Fix a tuple $\vec{x}$ of reals. The argument that follows is uniform in $\vec{x}$. Suppose that $R(\vec{x}, \omega_1)$ holds.

Say that M is a *candidate* iff it is a transitive model of ZFC^- such that $x, \omega_1 \in M$. By reflection, using that R is Σ_1, it follows there is a *small candidate* M such that $M \models R(\vec{x}, \omega_1)$, where a *small candidate* is a candidate of size ω_1. Thus, there is a set $A \subseteq \omega_1$ coding a small candidate that models $R(\vec{x}, \omega_1)$. Here, that A codes M_A means that, viewing A as a binary relation on ω_1 (via Gödel pairing), we have that (ω_1, A) is isomorphic to $(M_A, \in)$. Note that in any transitive model of ZFC^- that contains A as an element, M_A is also an element.

To say that A codes such a small candidate is equivalent to saying that there is an ordinal β such that $A \in L_\beta[A, \vec{x}]$, $L_\beta[A, \vec{x}] \models \mathsf{ZFC}^-$, and

$$L_\beta[A, \vec{x}] \models \psi(\vec{x}, A),$$

where ψ is a statement indicating that the model coded by A satisfies $R(\vec{x}, \omega_1)$.

Letting β_0 be the least such β, we have that β_0 is of size ω_1 and there is a club C of ordinals $\alpha < \omega_1$ and a sequence $(M_\alpha \mid \alpha \in C)$ of countable models such that

$$\forall \alpha \in C\ (M_\alpha \prec L_{\beta_0}[A, \vec{x}] \text{ and } M_\alpha \cap \omega_1 = \alpha).$$

Let $Y \subseteq \omega_1$ code (C, A) in the sense that $\mathrm{Odd}(Y) = A$ and if $Y_0 = \mathrm{Even}(Y)$ and $\{c_\alpha \mid \alpha < \omega_1\}$ is the increasing enumeration of the elements of C, then:

- $Y_0 \cap \omega$ codes a well-ordering of type c_0.
- $Y_0 \cap [\omega, c_0) = \emptyset$.
- For all α, $Y_0 \cap [c_\alpha, c_\alpha + \omega)$ codes a well-ordering of type $c_{\alpha+1}$.
- For all α, $Y_0 \cap [c_\alpha + \omega, c_{\alpha+1}) = 0$.

Note that the following statement $(*)$ holds:

Whenever $\mathscr{M}$ is a countable transitive model of ZFC^- such that $Y \cap \omega_1^{\mathscr{M}} \in \mathscr{M}$, $\omega_1^{\mathscr{M}} = (\omega_1^L)^{\mathscr{M}}$, and $\vec{x} \in \mathscr{M}$, then

$$\mathscr{M} \models R(\vec{x}, \omega_1^{\mathscr{M}}).$$

This is because for any such $\mathscr{M}$, $\delta = \omega_1^{\mathscr{M}}$ belongs to C, $A \cap \delta \in \mathscr{M}$, and $(\delta, A \cap \delta)$ is isomorphic to a transitive model N; moreover, since N is obtained via the transitive collapse of $(\delta, A \cap \delta)$, $N \in \mathscr{M}$. As N satisfies $R(\vec{x}, \omega_1^{\mathscr{M}})$ and $R(\vec{x}, z)$ is a Σ_1 formula, so does $\mathscr{M}$.

Let $\vec{r}$ be the canonical L-sequence of ω_1 many almost disjoint reals defined by setting $\vec{r} = (r_\alpha \mid \alpha < \omega_1)$ where the set $r_\alpha \subseteq \omega$ consists of those numbers that code a finite initial segment of the α-th real in the natural well-ordering of L. These sets r_α have pairwise finite intersection.

Let $\mathbb{P}$ be the almost disjoint coding forcing that codes Y as a real r relative to $\vec{r}$. Recall that conditions in $\mathbb{P}$ are pairs (s, a) where s is a finite subset of ω and a is a finite subset of $\{r_\beta \mid \beta \in Y\}$. Extension is defined by: $(s, a) \leq (t, b)$ iff s end-extends t, a contains b as a subset, and $s \setminus t$ is disjoint from each element of b.

This forcing is ccc because any two conditions with the same first component are compatible and there are only countably many first components. The generic produces a subset z of ω such that, for all countable β, z is almost disjoint from r_β exactly if β belongs to Y. Then the following property $(**)$ holds:

For any countable transitive model M of ZFC^- such that $z, \vec{x} \in M$ and $M \models \omega_1 = \omega_1^L$, we have that $M \models R(\vec{x}, \omega_1^M)$.

This is because any such M can reconstruct $Y \cap \omega_1^M$ and so we can apply $(*)$.

Since we are assuming MA_{ω_1}, there is in V a real z as above.

This shows that there is a Σ^1_3 statement $\varphi_R(\vec{x})$ (namely, the assertion that there is a real z such that $(**)$ holds) such that $\varphi(\vec{x})$ holds whenever $R(\vec{x}, \omega_1)$ does.

Conversely, If $\varphi_R(\vec{x})$ holds as witnessed by the real z, then $(**)$ holds without the restriction that M be countable, by reflection. But then $R(\vec{x}, \omega_1)$ holds.

This completes the proof of Theorem 4.

But Theorem 1 follows at once as well, noticing that the argument from Caicedo-Veličković [5] shows that, under the assumption of $\mathsf{BPFA} + \omega_1 = \omega_1^L$, there is a Σ_1 well-ordering of $H(\omega_2)$ in ω_1 as a parameter, since any transitive model M of ZFC^- that computes ω_1 correctly would be able to compute correctly the L-least C-sequence $\vec{C}$, and this is also a C-sequence in V and M.

Remark 5. The argument above can be generalized, as long as there is no inner model with ω many strong cardinals, since in this case K exists and only has finitely many strong cardinals, see Caicedo-Schindler [4, Theorem 2]. For example: Suppose that $0^\P$ does not exist and that $\omega_1 = \omega_1^K$. Then the set of codes for locally countable initial segments of K is Π^1_2, see Zeman [17].

The argument above gives us in this case that, if MA holds, then Σ_1 properties of reals with parameter ω_1 are equivalent to Σ^1_4 properties. It then follows that if in fact BPFA holds, then there is a Σ^1_4 well-ordering of the reals.

See Friedman-Schindler [8, Corollary 2.3] for the corresponding computation of codes in the presence of only finitely many strong cardinals in K, from which projective well-orderings of the reals can be extracted by our argument if $\omega_1 = \omega_1^K$ and BPFA holds.

§3. Coding relations in a Σ^1_4 fashion. In what follows, we identify definable relations with their definitions. Given a relation R' on $\mathbb{R}$ definable (using ground model parameters), and an iteration $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \kappa \rangle$, say that a relation R on $\mathbb{R}$ in the extension by $\mathbb{P}_\kappa$ is *locally certified* by R' with respect to $\vec{\mathbb{P}}$ iff the following two conditions hold:

1. Uniformly in the ground model, whenever G is $\mathbb{P}_\kappa$-generic, for each tuple $\vec{r}$ of reals of $V[G]$ (of the appropriate length) such that $V[G] \models R(\vec{r})$, we can identify an intermediate stage α such that $\vec{r}$ already belongs to the α-th intermediate model $V[G_\alpha]$, and $V[G_\alpha] \models R'(\vec{r})$.
2. R in $V[G]$ is the relation given by $R(\vec{r})$ iff $V[G_\alpha] \models R'(\vec{r})$ for some α as above.

The goal of this section is to show that, working over L in the presence of a reflecting cardinal κ, given a definable R', there is a countable support iteration $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \kappa \rangle$ forcing BPFA such that, if G is $\mathbb{P}_\kappa$-generic over V and $R \in V[G]$ is locally certified by R' with respect to $\vec{\mathbb{P}}$, then R is Σ^1_4.

(Note that $R(\vec{r})$ does not necessarily imply $R'(\vec{r})$ in $V[G]$.)

Fix R', that (to ease notation) we assume binary. The iteration we consider enhances the standard (Goldstern-Shelah) iteration that forces BPFA, by including stages at which certain trees are *specialized*, following a method of Baumgartner [2], and at which "Π^1_2 witnesses" to these specializations are added, following the method of David. To prevent the witnessing of BPFA from damaging the codings, we are forced to concentrate the iteration on stages α that are Σ^L_2 *stable*, i.e., such that L_α is Σ_2-elementary in L. Unfortunately, this forces us to also introduce Π^1_2 witnesses to failures of Σ^L_2 stability. These last witnesses lead us to a Σ^1_4 definition of the relation R, rather than Σ^1_3.

Typical examples of these relations R are well-orderings of $\mathbb{R}$. It was in this form that we originally found this result (BPFA is equiconsistent with the additional requirement that there is a Σ^1_4 well-ordering of the reals), and the stronger Theorem 1 uses several ideas of the original argument.

For reasons having to do with the forcings that add *localising* witnesses, the factors in the iteration will not be proper but only $\mathcal{S}$-*proper*, in the sense described below. After reviewing the notion of $\mathcal{S}$-properness, we prove a combinatorial lemma that will be used to carry out the coding.

Definition 6. Say that a class $\mathcal{S}$ is *closed under truncation* iff for all regular uncountable cardinals θ and all $x \in S$, we have that $x \cap H(\theta) \in S$.

A class $\mathcal{S}$ is *everywhere stationary* iff S is closed under truncation, and its intersection with $[H(\theta)]^\omega$ is stationary for all uncountable regular cardinals θ.

Suppose that $\mathcal{S}$ is everywhere stationary. A partial order $\mathbb{P}$ is $\mathcal{S}$-*proper* iff for all regular cardinals $\theta > \omega_1$ such that $\mathbb{P} \in H(\theta)$, there is a club of countable elementary substructures x of $H(\theta)$ with the property that if $x \in \mathcal{S}$ and $p \in \mathbb{P} \cap x$, then there is $q \leq p$ in $\mathbb{P}$ which forces the generic to intersect $D \cap x$ for any $D \in x$ that is dense in $\mathbb{P}$.

$\mathcal{S}$-properness is a Σ_2 notion (in the predicate $\mathcal{S}$), as "all regular cardinals θ" can be replaced by "the least regular cardinal θ" in the above definition. This is because if $\theta > \omega_1$ is the least regular cardinal such that $\mathbb{P} \in H(\theta)$, C witnesses the desired property for θ, and $\tau > \theta$ is regular, then (using closure under truncation) we have that

$$C^* = \{x : x \cap H(\theta) \in C\}$$

witnesses the desired property for τ.

Just as with the usual notion of properness, $\mathcal{S}$-proper forcing notions preserve ω_1, and $\mathcal{S}$-properness is preserved under countable support iterations (see Shelah [15]).

We make essential use of the following lemma. For β a regular uncountable cardinal, let $T(\beta)$ be the tree $(\beta^+)^{<\beta}$ of sequences through β^+ of length less than β.

Lemma 7. *Assume $V = L$ and that $\beta > \omega_1$ is regular. Let $\mathcal{S}$ be an everywhere stationary class. Suppose that $\mathbb{Q}$ is an $\mathcal{S}$-proper forcing, that $|\mathbb{Q}| < \beta$, and that G is $\mathbb{Q}$-generic over L. Then*:

1. *$T(\beta)$, viewed as a forcing, is $\mathcal{S}$-proper in $L[G]$.*
2. *There is a proper forcing $\mathbb{R}$ in $L[G]$ of size β^{++} that destroys the $\mathcal{S}$-properness of $T(\beta)$; in fact, if H is $\mathbb{R}$-generic over $L[G]$, then in any ω_1-preserving outer model of $L[G][H]$ there is no branch through $T(\beta)$ which is $T(\beta)$-generic over L.*

Proof. (1) Since $\mathbb{Q} * \check{T(\beta)} \equiv \mathbb{Q} \times T(\beta)$, it suffices to show that $\mathbb{Q}$ is $\mathcal{S}$-proper in $T(\beta)$-generic extensions of L. But the forcing $T(\beta)$ is β-closed and therefore does not add subsets of $\max\{|\mathbb{Q}|, \omega_1\}$; it follows that any witness to the $\mathcal{S}$-properness of $\mathbb{Q}$ in L is still a witness to its $\mathcal{S}$-properness in any $T(\beta)$-generic extension of L.

(2) First add β^{++} Cohen reals with a finite support product over $L[G]$, producing $L[G][H_0]$. Then Lévy collapse β^{++} to ω_1 with countable conditions, producing $L[G][H_0][H_1]$. As ccc and ω-closed forcings are proper, this is a proper forcing extension of $L[G]$.

Note that (as originally shown by Silver) in $L[G][H_0][H_1]$, any β-branch through $T(\beta)$ in fact belongs to $L[G][H_0]$: Otherwise we choose an $L[G][H_0]$-name $\dot{b}$ for the new branch and build a binary ω-tree U of conditions in the Lévy collapse, each branch of which has a lower bound, such that distinct branches force different interpretations of the name $\dot{b}$. It follows that in $L[G][H_0]$, $T(\beta)$ has $2^{\aleph_0} = \beta^{++}$ nodes on a fixed level, which is impossible because GCH holds in L.

Thus the tree $T(\beta)$ has at most ω_1-many branches in $L[G][H_0][H_1]$, none of which contains ordinals cofinal in β^+ and therefore none of which is $T(\beta)$-generic over L. Also, every node of $T(\beta)$ belongs to a β-branch.

Now we use Baumgartner's general method of "specializing a tree off a small set of branches".

Fact 8. *If T is a tree of height ω_1 with at most $\aleph_1$ cofinal branches (and every node of T belongs to a cofinal branch of T) then there is a ccc forcing $\mathbb{P}$ such that if G is $\mathbb{P}$-generic over V then in any ω_1-preserving outer model of $V[G]$, all cofinal branches through T belong to V.*

Proof. We outline the argument and refer the reader to Baumgartner [2] for details.

List the branches as $(b_i \mid i < \omega_1)$ and write T as the disjoint union of $b_i(x_i)$, where the x_i are distinct nodes of the tree chosen so that each x_i is a node on b_i and $b_i(x_i)$ denotes the tail of b_i starting at x_i. Now force to add a function f with finite conditions from $\{x_i \mid i < \omega_1\}$ into ω such that if x_i is below x_j in T then $f(x_i)$ is different from $f(x_j)$. Baumgartner [2] shows that this forcing is ccc. Now if b is a cofinal branch through T distinct from all the branches b_i in an ω_1-preserving outer model of $V[f]$, then b must intersect uncountably many of the branches $b_i(x_i)$ and therefore contains uncountably many nodes x_i. But then the numbers $f(x_i)$ are distinct for these uncountably many nodes x_i, contradicting the fact that f maps into ω.

This completes the proof of Fact 8. ⊣

Now use Fact 8 to create a ccc extension $L[G][H_0][H_1][H_2]$ of $L[G][H_0][H_1]$ to ensure that $T(\beta)$ (viewed as a tree of height ω_1 using a cofinal ω_1-sequence through $(\beta^+)^L$) will have no new branches in any ω_1-preserving outer model. As no β-branch through $T(\beta)$ in $L[G][H_0]$ is $T(\beta)$-generic over L and all cofinal branches through $T(\beta)$ in any ω_1-preserving outer model of $L[G][H_0][H_1][H_2] = L[G][H]$ belong to $L[G][H_0]$, we are done.

This completes the proof of Lemma 7. ⊣

We now begin the proof of the coding result.

Assume $V = L$ and let κ be reflecting. Fix an appropriate bookkeeping function $f : \kappa \to H(\kappa)$ (so that f "guesses" every object in $H(\kappa)$ stationarily often). We will use f throughout the argument to select certain objects. We use a countable support iteration of length κ. The factors in our iteration will be $\mathcal{S}$-proper for a suitable everywhere stationary class $\mathcal{S}$, that we now proceed to describe.

Suppose that θ is regular and uncountable, and that x is a countable elementary substructure of L_θ. Let $(x, \in)$ be isomorphic to L_α. We say that x *collapses nicely* iff for all $\beta \geq \alpha$, if L_β is a model of ZFC^- and $x \cap \omega_1$ is a cardinal in L_β, then every cardinal of L_α is also a cardinal of L_β.

Let $\mathcal{S}$ be the class of all x in L which collapse nicely.

Lemma 9. *$\mathcal{S}$ is everywhere stationary.*

Proof. Let θ be regular and uncountable, and let $C \subseteq [L_\theta]^\omega$ be club, so $C \in L_{\theta^+}$. Let x be the least elementary substructure of L_{θ^+} that contains C as an element. Then $x \cap L_\theta \in C$. Let L_α be the transitive collapse of $(x, \in)$. Then there is an $L_{\alpha+1}$-definable injection from L_α into ω and, therefore, there is *no* $\beta > \alpha$ such that $L_\beta \models \mathsf{ZFC}^-$ and $x \cap \omega_1$ is a cardinal of L_β. It follows that $x \in S$ and therefore $x \in S \cap C$. Since S is clearly closed under truncation, we are done. ⊣

Let C enumerate the closed unbounded subset of κ consisting of those α such that L_α is Σ_2-elementary in L_κ. (As κ is regular, C is indeed unbounded in κ.) We perform an $\mathcal{S}$-proper iteration of length κ with countable support which is nontrivial at stages α in C. The iteration $\mathbb{P}_\alpha * \dot{\mathbb{Q}}(\alpha)$ up to and including stage α will belong to L_β where β is the least element of C greater than α. In particular, $|\mathbb{P}_\alpha| < \kappa$ for each $\alpha < \kappa$, and therefore κ remains reflecting throughout the iteration.

Suppose that α belongs to C. We proceed to describe the forcing $\mathbb{Q}(\alpha)$ as a six-step iteration $\mathbb{Q}^0(\alpha) * \dot{\mathbb{Q}}^1(\alpha) * \dot{\mathbb{Q}}^2(\alpha) * \dot{\mathbb{Q}}^3(\alpha) * \dot{\mathbb{Q}}^4(\alpha) * \dot{\mathbb{Q}}^5(\alpha)$. As usual, the $\mathbb{Q}^i(\alpha)$ are trivial, unless the bookkeeping function f at stage α gives us an object of the appropriate kind, as specified in each instance below.

3.1. $\mathbb{Q}^0(\alpha)$**.** Inductively, $\mathbb{P}_\alpha$ has size at most $(\alpha^+)^L$. By Lemma 7, we know that the forcing $T(\beta)$, consisting of $(< \beta)$-sequences through β^+, is $\mathcal{S}$-proper in $L[G_\alpha]$ when β is regular and at least $(\alpha^{++++})^L$. In addition, there is a forcing $\mathbb{R}(\beta)$ of size β^{++} in $L[G_\alpha]$ which guarantees that there is no $T(\beta)$-generic over L.

Now let α_n be $(\alpha^{+4(n+1)})^L$ for each finite n, and let $T(n)$, $\mathbb{R}(n)$ denote $T(\alpha_n)$, $\mathbb{R}(\alpha_n)$. Then both $T(n)$ and $\mathbb{R}(n)$ are $\mathcal{S}$-proper in any extension of $L[G_\alpha]$ obtained by forcing with $U(0) * U(1) * \cdots * U(n-1)$ where each $U(i)$ is either $T(i)$ or $\mathbb{R}(i)$.

Let R_α denote R' from the point of view of $L[G_\alpha]$ and let $(x_\alpha, y_\alpha) \in R_\alpha$ be the pair of reals in $L[G_\alpha]$ provided by the bookkeeping function (which guarantees that any pair (x, y) of reals which appears in the iteration is of the form (x_α, y_α) for some α).

Now take $\mathbb{Q}^0(\alpha)$ to be the (fully supported) ω-iteration $U(0) * U(1) * \cdots$ where $U(n)$ equals $T(n)$ if n belongs to $x_\alpha * y_\alpha$ (the join of x_α and y_α) and equals $\mathbb{R}(n)$ otherwise. This is an $\mathcal{S}$-proper forcing and $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0(\alpha)$ belongs to L_β, where β is the least element of C greater than α.

3.2. $\mathbb{Q}^1(\alpha)$**.** Now we consider the Σ_1 sentence with parameter from $L[G_\alpha] \cap \mathscr{P}(\omega_1)$, provided by the bookkeeping function (which ensures that all Σ_1 sentences with parameter from the final $\mathscr{P}(\omega_1)$ will be considered at some stage $\alpha < \kappa$ in C).

Ask of this sentence whether it holds in an $\mathcal{S}$-proper forcing extension of $L[G_\alpha][H^0]$, where H^0 is our $\mathbb{Q}^0(\alpha)$-generic. If so, then as κ is reflecting in $L[G_\alpha][H^0]$, there is such an $\mathcal{S}$-proper forcing in $L_\kappa[G_\alpha][H^0]$, and also the witness to the Σ_1 sentence can be assumed to have a name in $L_\kappa[G_\alpha][H^0]$. Let β be the least element of C greater than α; then as L_β is Σ_2-elementary in L_κ, it follows that $L_\beta[G_\alpha][H^0]$ is Σ_2-elementary in $L_\kappa[G_\alpha][H^0]$. Thus we can choose our $\mathcal{S}$-proper forcing $\mathbb{Q}^1(\alpha)$ witnessing the Σ_1 sentence to be an element of $L_\beta[G_\alpha][H^0]$, necessary to satisfy the requirement that $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^1(\alpha)$ belong to L_β. Let H^1 denote the generic for $\mathbb{Q}^1(\alpha)$.

3.3. $\mathbb{Q}^2(\alpha)$**.** The forcing $\mathbb{Q}^2(\alpha)$ has the form $\mathbb{D}_1(\alpha) * \mathbb{D}_2(\alpha)$. To see this, begin by noticing that if $\tau < \kappa$ is a strong limit singular cardinal of uncountable cofinality, then $2^\tau = \tau^+$ and $\tau^+ = (\tau^+)^L$. Otherwise, covering fails and $0^\sharp$ exists. There is therefore a set $A' \subseteq \tau^+$ such that $H(\tau^+) = L_{\tau^+}[A']$. Let $\mathbb{D}_1(\alpha) = \mathtt{Col}(\omega_1, \tau)$. This is a countably closed forcing and it adds a subset H^1 of ω_1 such that

$$H(\omega_2) = L_{\omega_2}[A', H^1]$$

holds in the extension.

We can by further countably closed forcing arrange that there is a subset X_α of ω_1 such that

$$H(\omega_2) = L_{\omega_2}[X_\alpha].$$

This is well-known (see, e.g., Schindler [13, Claims 1,2]). For example, we can pick a sequence of almost disjoint subsets of τ in L. This gives us a sequence $\mathscr{A}$ of almost disjoint subsets of ω_1 via a bijection between ω_1 and τ. We can then force to code $A' \oplus A''$ as a subset H^2 of ω_1 using the sequence $\mathscr{A}$. Call $\mathbb{D}_2(\alpha)$ the corresponding forcing notion.

We thus have that the resulting extension $L[G_\alpha][H^0][H^1][H^2]$ is of the form $L[X_\alpha]$ where X_α is a subset of ω_1 which codes the ordinal α as well as the generic $H^0 * H^1 * H^2$.

Then we have:

($*$) If $M = L_\delta[X_\alpha]$ is a transitive model of ZFC^-, then $(\alpha^{+\omega})^L$ is an ordinal of M, and in M there is a branch through $T((\alpha^{+4(n+1)})^L)$ whose ordinals are cofinal in $(\alpha^{+4(n+1)})^L$ iff n belongs to $x_\alpha * y_\alpha$.

3.4. $\mathbb{Q}^3(\alpha)$**.** The purpose of the forcing $\mathbb{Q}^3(\alpha)$ is to add $Y_\alpha \subseteq \omega_1$ that "localizes" property $(*)$ in the following sense:

> $(**)$ For any $\gamma < \omega_1$ and countable transitive model M of ZFC^- containing $Y_\alpha \cap \gamma$ as an element: If $\gamma = \omega_1^M = (\omega_1^L)^M$ then $\mathrm{Odd}(\mathrm{Even}(Y_\alpha \cap \gamma))$ codes an L-cardinal $\bar{\alpha}$ of M such that there is a branch through the $T((\bar{\alpha}^{+4(n+1)})^L)$ of M whose ordinals are cofinal in the $(\bar{\alpha}^{+4(n+1)})^L$ of M iff n belongs to $x_\alpha * y_\alpha$.

We now describe the forcing $\mathbb{Q}^3(\alpha)$ for adding the witness Y_α to $(**)$. A condition in $\mathbb{Q}^3(\alpha)$ is an ω_1-Cohen condition $r : |r| \to 2$ in $L[X_\alpha]$ with the following properties:

1. The domain $|r|$ of r is a countable limit ordinal.
2. $X_\alpha \cap |r|$ is the even part of the even part of r, i.e., for $\gamma < |r|$, γ belongs to X_α iff $r(4\gamma) = 1$.
3. $(**)_r$ holds. This is the statement that $(**)$ holds for all limit $\gamma \le |r|$ with $Y_\alpha \cap \gamma$ replaced by $r \restriction \gamma$.

Lemma 10. *$\mathbb{Q}^3(\alpha)$ is $\mathcal{S}$-proper.*

Proof. First note that we have the following *extendibility property*: Given r and a countable limit γ greater than $|r|$, we can extend r to r^* of length γ.

This is because we can take the odd part of r^* on the interval $[|r|, |r| + \omega)$ to code γ and to consist only of 0s on $[|r| + \omega, \gamma)$; then there are no new instances of requirement (3) for being a condition to check because no transitive model of ZFC^- containing $r^* \restriction |r| + \omega$ can have its ω_1 in the interval $(|r|, \gamma]$.

Now in $L[X_\alpha]$ let θ be large and regular, let M be countable and elementary in $H(\theta)$ with $M \cap L$ in $\mathcal{S}$ and let r belong to $\mathbb{Q}^3(\alpha) \cap M$. Successively extend r to $r = r_0 \ge r_1 \ge \cdots$ in M so that if D in M is dense on $\mathbb{Q}^3(\alpha)$ then for some k, r_k meets D. (In particular, r_k forces the $\mathbb{Q}^3(\alpha)$-generic to meet D in a condition belonging to M.) By extendibility, $\sup r_k$ converges to $\delta := M \cap \omega_1$.

We want to show that the conditions r_k admit the lower bound $r_\omega = \bigcup_k r_k$. For this, it suffices to verify property $(**)_{r_\omega}$ when $\gamma = \delta$, i.e.:

> $(***)$ For any countable transitive model N of ZFC^- containing r_ω as an element: If $\delta = \omega_1^N = (\omega_1^L)^N$ then $\mathrm{Odd}(\mathrm{Even}(r_\omega))$ codes an L-cardinal $\bar{\alpha}$ of M such that there is a branch through the $T((\bar{\alpha}^{+4(n+1)})^L)$ of M whose ordinals are cofinal in the $(\bar{\alpha}^{+4(n+1)})^L$ of M iff n belongs to $x_\alpha * y_\alpha$.

Let $\bar{M} = L_{\bar{\theta}}[X_\alpha \cap \delta]$ be the transitive collapse of M, where a is sent to $\bar{a}$ under the transitive collapse map. As X_α codes the generic $G_\alpha * H^0 * H^1 * H^2$, it ensures that in $L_\theta[X_\alpha]$ there is a branch through $T((\alpha^{+4(n+1)})^L)$ whose ordinals are cofinal in $(\alpha^{+4(n+1)})^L$ iff n belongs to $x_\alpha * y_\alpha$. By elementarity, in $\bar{M}$ there is a branch through the $T((\alpha^{+4(n+1)})^L)$ of $\bar{M}$ whose ordinals are cofinal in the $(\alpha^{+4(n+1)})^L$ of $\bar{M}$ iff n belongs to $x_\alpha * y_\alpha$.

Now if $\bar{N}$ is any countable transitive model of ZFC^- containing r_ω as an element such that $\omega_1^{\bar{N}} = \delta$, $\bar{N}$ also contains $X_\omega \cap \delta$ as an element and as $M \cap L = L^M$ collapses nicely, the $(\alpha^{+4(n+1)})^L$, $T((\alpha^{+4(n+1)})^L)$ of $\bar{M}$ are equal to those of $\bar{N}$. It follows that also in $\bar{N}$, there is a branch through the $T((\alpha^{+4(n+1)})^L)$ of $\bar{N}$ whose ordinals are cofinal in the $(\alpha^{+4(n+1)})^L$ of $\bar{N}$ iff n belongs to $x_\alpha * y_\alpha$, establishing $(***)$. ⊣

3.5. $\mathbb{Q}^4(\alpha)$**.** We next code the $\mathbb{Q}^3(\alpha)$-generic Y_α by a real using $\mathbb{Q}^4(\alpha)$, the ccc almost disjoint coding with finite conditions denoted by $\mathbb{P}$ in the proof of Theorem 4.

3.6. $\mathbb{Q}^5(\alpha)$**.** To complete stage α of the iteration we apply a forcing $\mathbb{Q}^5(\alpha)$ introducing Π^1_2 witnesses to failures of Σ^L_2 stability.

Let z_α be the real in $L[G_\alpha]$ provided by the bookkeeping function (so that each real that appears anywhere in the iteration is equal to z_α for some $\alpha \in C$).

We say that z_α is a *coding witness for* $R(x, y)$ (where x, y are reals in $L[G_\alpha]$) iff we have:

$(*)_{z_\alpha,x,y}$ For any countable transitive model M of ZFC^- that contains z_α, x, y as elements and such that $\omega_1^M = (\omega_1^L)^M$, z_α codes in M some $\bar{\alpha}$, an L-cardinal of M, such that $T((\bar{\alpha}^{+4(n+1)})^L)$ has a branch whose ordinals are cofinal in $(\bar{\alpha}^{+4(n+1)})^L$ iff n belongs to $x * y$.

Note that by reflection, $(*)_{z_\alpha,x,y}$ holds without the restriction that M be countable. Let δ be the L-cardinal witnessing $(*)_{z_\alpha,x,y}$ for the model

$$M = L_{\kappa^+}[G_\alpha][H^0][H^1][H^2][H^3][H^4],$$

where H^i is the generic for $\mathbb{Q}^i(\alpha)$. Then if δ is *not* Σ^L_2 stable, the forcing $\mathbb{Q}^5(\alpha)$ introduces a real w_α such that:

$(****)_{z_\alpha,w_\alpha}$ For all countable transitive models M of ZFC^- containing z_α, w_α as elements and such that $\omega_1^M = (\omega_1^L)^M$, w_α codes in M some $\bar{\beta}$, an L-cardinal of M, such that $L_{\bar{\alpha}}$, where $\bar{\alpha}$ is the L-cardinal of M coded by z_α, is not Σ_2-elementary in $L_{\bar{\beta}}$.

The forcing $\mathbb{Q}^5(\alpha)$ is defined analogously to the two-step iteration $\mathbb{Q}^3(\alpha) * \mathbb{Q}^4(\alpha)$, and like that forcing, it is $\mathcal{S}$-proper.

This completes stage α of the iteration.

3.7. R **is** Σ^1_4**.** The iteration so defined is $\mathcal{S}$-proper, forces κ to be at most ω_2, and is κ-cc. It follows that $\kappa = \omega_2$ in the generic extension $L[G]$, and the standard argument shows that BPFA (indeed, the bounded forcing axiom for $\mathcal{S}$-proper forcings) holds there. Note that (by construction) R is locally certified by R' with respect to this iteration.

To verify that R is Σ^1_4, say that a real z is a *good coding witness for* $R(x, y)$ iff it is a coding witness for $R(x, y)$, and there is *no* w witnessing the failure of the Σ^L_2 stability of the L-cardinal coded by z, i.e., there is no real w such that $(****)_{z,w}$ holds.

The set of good witnesses is Π^1_3. Thus R is definable in $L[G]$ by:

$R(x, y)$ iff for some α in C, $(x, y) = (x^G_\alpha, y^G_\alpha)$ iff there exists a good coding witness for $R(x, y)$.

This completes the proof.

§4. Open questions. We close the paper with some natural problems suggested by the results above:

1. In Theorem 1, can the hypothesis $\omega_1 = \omega_1^L$ be weakened to $0^\sharp$ does not exist?
2. Is $\mathsf{MA} + \omega_1 = \omega_1^L$ consistent with the *nonexistence* of a projective well-ordering of the reals?
3. Does the existence of a $\Delta_1(A)$ well-ordering of $\mathbb{R}$ (for some parameter $A \in H(\omega_2)$) follow from the version of the bounded forcing axiom for posets of the from σ-closed$*$ccc?

REFERENCES

[1] JOAN BAGARIA, *Bounded forcing axioms as principles of generic absoluteness*, ***Archive for Mathematical Logic***, vol. 39 (2000), no. 6, pp. 393–401.

[2] JAMES BAUMGARTNER, *Applications of the proper forcing axiom*, ***Handbook of set-theoretic topology*** (Kenneth Kunen and Jerry Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 913–959.

[3] ANDRÉS E. CAICEDO, *Projective well-orderings and bounded forcing axioms*, this JOURNAL, vol. 70 (2005), no. 2, pp. 557–572.

[4] ANDRÉS E. CAICEDO and RALF SCHINDLER, *Projective well-orderings of the reals*, ***Archive for Mathematical Logic***, vol. 45 (2006), pp. 783–793.

[5] ANDRÉS E. CAICEDO and BOBAN VELIČKOVIĆ, *The bounded proper forcing axiom and well-orderings of the reals*, ***Mathematical Research Letters***, vol. 13 (2006), no. 2–3, pp. 393–408.

[6] SY-D. FRIEDMAN, *David's trick*, ***Sets and proofs (Leeds, 1997)*** (Barry Cooper and John Truss, editors), London Mathematical Society Lecture Note Series, vol. 258, Cambridge University Press, Cambridge, 1999, pp. 67–71.

[7] ———, ***Fine structure and class forcing***, de Gruyter Series in Logic and its Applications, vol. 3, Walter de Gruyter, Berlin, 2000.

[8] SY-D. FRIEDMAN and RALF SCHINDLER, *Universally Baire sets and definable well-orderings of the reals*, this JOURNAL, vol. 68 (2003), no. 4, pp. 1065–1081.

[9] MARTIN GOLDSTERN and SAHARON SHELAH, *The bounded proper forcing axiom*, this JOURNAL, vol. 60 (1995), no. 1, pp. 58–73.

[10] GREG HJORTH, *The size of the ordinal* u_2, ***Journal of the London Mathematical Society (2)***, vol. 52 (1995), no. 3, pp. 417–433.

[11] DONALD A. MARTIN and ROBERT M. SOLOVAY, *Internal Cohen extensions*, ***Annals of Mathematical Logic***, vol. 2 (1970), no. 2, pp. 143–178.

[12] JUSTIN MOORE, *Set mapping reflection*, ***Journal of Mathematical Logic***, vol. 5 (2005), no. 1, pp. 87–98.

[13] RALF SCHINDLER, *Coding into K by reasonable forcing*, ***Transactions of the American Mathematical Society***, vol. 353 (2000), pp. 479–489.

[14] ———, *Proper forcing and remarkable cardinals II*, this JOURNAL, vol. 66 (2001), no. 3, pp. 1481–1492.

[15] SAHARON SHELAH, ***Proper and improper forcing***, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.

[16] SAHARON SHELAH and LEE J. STANLEY, *Coding and reshaping when there are no sharps*, ***Set theory of the continuum (Berkeley, CA, 1989)***, Math. Sci. Res. Inst. Publ., vol. 26, Springer, New York, 1992, pp. 407–416.

[17] MARTIN ZEMAN, ***Inner models and large cardinals***, de Gruyter Series in Logic and its Applications, vol. 5, Walter de Gruyter, Berlin, 2002.

BOISE STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS—1910 UNIVERSITY DRIVE
BOISE, ID 83725, USA
URL: http://math.boisestate.edu/~caicedo/
E-mail: caicedo@math.boisestate.edu

KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
UNIVERSITY OF VIENNA
VIENNA, AUSTRIA
URL: http://www.logic.univie.ac.at/~sdf/
E-mail: sdf@logic.univie.ac.at