Simply definable well-orderings of the reals

by

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Abstract

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In this thesis we explore the problem of obtaining simply definable well-orderings of the reals together with additional combinatorial structure on the continuum. Specifically:

- We show that the following is consistent, without any restrictions in the large cardinal structure of the universe: "$\mathfrak{c}$ is real-valued measurable and there is a definable Δ^2_2-well-ordering of the reals". More precisely: If there is a measurable, and GCH holds below it, then there is a forcing extension satisfying the statement.

- We present a similar argument, due to Woodin, that when applied to $L[\mu]$ produces a forcing extension where $\mathfrak{c}$ is real-valued measurable, and there is a Δ^2_1-well-ordering of $\mathbb{R}$. The best result along these lines, due to Woodin, is that under appropriate large cardinal hypothesis, "$\mathfrak{c}$ is real-valued measurable and there is a definable Δ^2_1-well-ordering of the reals" is Ω-consistent.

- We introduce a strengthening of real-valued measurability, called real-valued hugeness, which implies the existence of many real-valued measurable cardinals and by results of Woodin, the determinacy of strong pointclasses. For example, $\mathsf{AD}^{L(\mathbb{R}^\sharp)}$ holds. We show it is consistent that $\mathfrak{c}$ is real-valued huge, and present a result of Woodin proving that this property of $\mathfrak{c}$ contradicts the existence of any Δ^2_n-well-ordering of $\mathbb{R}$.

- We show that if there is no inner model with ω many strong cardinals, then there is a set forcing extension with a projective well-ordering of the reals (In fact, a $\underset{\sim}{\Sigma}^1_{n+3}$-

well-ordering, where n is the number of strong cardinals in K and, if $n = 0$ and V is not closed under sharps, then a $\underset{\sim}{\Sigma}^1_2$-one.)

- We show that if there are no inner models with Woodin cardinals, and V is a finestructural model for a strong cardinal with a measurable above, then there is a forcing extension where $\mathsf{SPFA}(\mathfrak{c}) + \mathsf{BSPFA}^{++} + \psi_{AC}$ hold and where the reals admit a Σ^1_5-well-ordering.

We also start an investigation into the structure of inner models M of GCH when a strong forcing axiom holds in V, and show:

- Without loss of generality, $\omega_1^M = \omega_1$. More precisely: If $\mathsf{PFA}(\mathfrak{c})$ holds, then there is an inner model N, $M \subseteq N$, such that $N \models \mathsf{GCH}$, $\omega_1 = \omega_1^N$, and ω_2^V is inaccessible in N iff it is inaccessible in M.
- Moreover, if $\mathsf{PFA}(\mathfrak{c})$ holds and ω_2^V is a successor cardinal in M, then $\omega_2^V = (\lambda^+)^M$, where $\mathrm{cf}(\lambda) = \omega$, and $\square^*_\lambda$ fails in M.
- In fact, whenever M is an inner model of GCH correctly computing $\aleph_1$ and such that $\aleph_2^V = (\lambda^+)^M$, where $\mathrm{cf}^V(\lambda) = \omega$, then
  - In M the approachability property fails at λ and there are no uniformly almost disjoint sequences for λ, in particular $\mathrm{cf}^M(\lambda) = \omega$.
  - V is not a weakly proper forcing extension of M, and there is no inner model of V that computes ω_2 correctly where CH holds. In particular, if $\mathsf{PFA}(\mathfrak{c})$ holds then there is a real r such that $M[r] \models \neg\mathsf{CH}$.
  - Furthermore, if $\omega_2^V = \aleph^M_{\omega+1}$, then $\mathsf{VWS}_{\aleph_\omega}$ fails in M, and $(S^{\lambda^+}_{\omega_1})^M =_{\mathrm{NS}_{\omega_2}} S^{\omega_2}_{\omega_1}$.

Professor John Steel
Dissertation Committee Co-chair

Professor W. Hugh Woodin
Dissertation Committee Co-chair

To the memory of Iya.

To my mom and mi Naji.

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But continue thou in the things which thou hast learned and hast been assured of, knowing of whom thou hast learned them;

2 Timothy 3:14

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ke bello essere un sapiente ki lo aveva mai detto

Baudolino
Umberto Eco
Milano, Bompiani, 2000; pp. 528

Chapter 1

Introduction

In this dissertation we explore problems related to the effect that combinatorial structure in the set-theoretic universe (beyond that already provided by the usual axioms) may have on the continuum or on sets at the level of the continuum. Mainly, we look at the definability of well-orderings of the reals in the presence of this additional combinatorial structure.

For notational conventions see Section 1.5.

1.1 Well-orderings of the reals

By the *reals* we mean here the members of any of the sets ω^ω, 2^ω, $[\omega]^\omega$, or even $\mathbb{R}$ or $[0,1]$. It is well known that ω^ω is homeomorphic to $\mathbb{R}\setminus\mathbb{Q}$ when endowed with the product topology. More importantly, all the spaces we have listed are standard Borel spaces and have an isomorphic Borel structure.

Definition 1.1. Let Σ be a σ-algebra on a space X containing all the singletons. A σ-finite measure λ on (X,Σ) is *continuous* iff $\lambda(\{x\})=0$ for any $x\in X$.

It is well known that for any continuous probability Borel measure λ on any standard Borel space X there is a Borel isomorphism $f:X\to[0,1]$ with $\lambda\circ f^{-1}=\mu$, Lebesgue measure on $[0,1]$.

We will abuse language a little bit. For example, we can therefore refer to "non-Lebesgue measurable subsets of reals", even if the standard Borel space we are considering is not literally the Euclidean $\mathbb{R}$ or the measure under discussion is not literally Lebesgue measure.

It is the working assumption of set-theorists that sets of reals whose existence is solely granted by AC are in general pathological, as opposed to those which can be "explicitly" defined. Recall ([K]) that the determinacy of sufficiently closed pointclasses implies that their members are Lebesgue measurable, have the perfect subset property, are Ramsey, etc. The following (folklore) result builds on a well known argument due to Sierpinski:

Theorem 1.2. *No well-ordering of a non-null set of reals is Lebesgue measurable. No well-ordering of a non-meager set of reals has the Baire property.*

Proof: We prove that whenever $S \subseteq \mathbb{R}$ is a non-null set, $\langle r_\alpha : \alpha < \lambda \rangle$ is a well-ordered enumeration of S, and $W = \{ (r_\alpha, r_\beta) : \alpha < \beta < \lambda \}$, then W is non-measurable. The argument admits dualization, giving the result for the property of Baire.

By contradiction, let λ be least such that there are a non-null set S and a well-ordered enumeration $\vec{r} = \langle r_\alpha : \alpha < \lambda \rangle$ of S such that W, defined as above, is measurable. For $\alpha < \lambda$, let $S_\alpha = \{ r_\beta : \beta < \alpha \}$ and $S^\alpha = \{ r_\beta : \alpha < \beta \}$. Let μ_n be n-dimensional Lebesgue measure. For $x \in \mathbb{R}$, by $r^{-1}x$ we mean the $\alpha < \lambda$ such that $x = r_\alpha$.

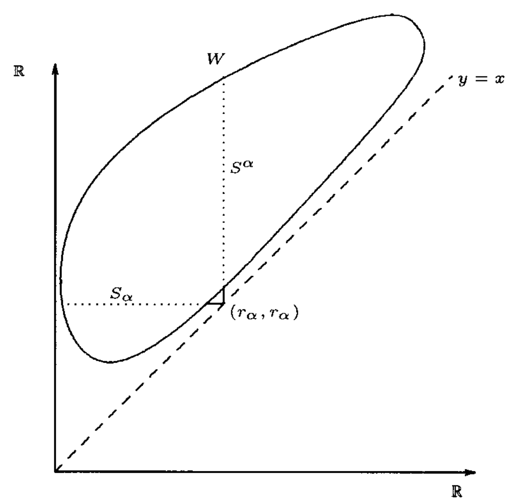

Figure 1.1: W is a well-ordering of $S \subseteq \mathbb{R}$

Notice that S is measurable: By Fubini's theorem, for almost all $x \in S$ and almost all $y \in S$, both $S^{r^{-1}x}$ and $S_{r^{-1}y}$ are measurable. Since $S = S_{\alpha+1} \cup S^\gamma$ for any $\alpha \leq \gamma < \lambda$, then S itself must be measurable.

Now we argue that for some $\gamma < \lambda$, S_γ is non-null and measurable. If so, we are done because $W_\gamma = W \cap (S_\gamma \times S_\gamma)$ is measurable and well-orders S_γ in order type $\gamma < \lambda$, contradicting the minimality of λ.

Since for almost all $y \in S$, $S_{r^{-1}y}$ is measurable, if no S_γ is as wanted then almost all of them have measure 0 and

$$\begin{aligned} 0 = \mu_2(W) &= \int_S \mu_1(S_{r^{-1}y})\, dy \\ &= \int_S \mu_1(S^{r^{-1}x})\, dx. \end{aligned}$$

But for almost all $x \in S$, $S^{r^{-1}x} = S \setminus (S_{r^{-1}x} \cup \{x\})$ has positive measure, and we have a contradiction. □

Remark 1.3. Notice that we are not using in any essential way that the measure under consideration is Lebesgue measure. Exactly the same proof works to show that if (X, Σ, ν) is any σ-finite continuous measure space, then no non-null subset of X admits a $\nu \times \nu$-measurable well-ordering. Hence, if $\nu(X) > 0$ then even if $\Sigma = \mathcal{P}(X)$ the completion of $\Sigma \times \Sigma$ cannot be $\mathcal{P}(X \times X)$.

This illustrates in part that large cardinal considerations make the problem of definability of well-orderings an interesting and difficult one. For example, mild large cardinal assumptions imply that every projective set of reals is determined and therefore Lebesgue measurable. Hence, no binary relation defined in second-order arithmetic can be a well-ordering of a non-trivial subset of $\mathbb{R}$. In fact under Projective Determinacy any projective well-ordering of a set of reals is countable. With this in mind, a natural target is that of Σ^2_n-definability. Before telling part of the story leading to this point, let us make it clear that all we strive for is the *consistency* of simply definable well-orderings without imposing bounds on the large cardinal structure of the universe. This can be understood, for example, as an argument establishing upper bounds for the complexity of well behaved pointclasses.

Recall that $\mathsf{HOD}_{\mathbb{R}}$ denotes the class of sets hereditarily ordinal definable using reals as parameters. For the definition of $\mathtt{Add}(\omega, \lambda)$ see Section 1.5; this is the standard forcing for adding λ Cohen reals. For the definition of $\mathtt{Random}_\lambda$ see Section 3.1, immediately after the statement of Theorem 3.6; this is the standard forcing for adding λ random reals.

Lemma 1.4. *Let G be $\mathbb{F}$-generic over V, where $\mathbb{F} = \mathtt{Add}(\omega, \lambda)$ or $\mathbb{F} = \mathtt{Random}_\lambda$, $\lambda \geq \omega_1$. Let $\mathbb{R} = \mathbb{R}^{V[G]}$. Then, in $V[G]$, there is $\mathbb{R}_0 \subset \mathbb{R}$ and a nontrivial elementary embedding $j : \mathsf{HOD}_{\mathbb{R}_0} \xrightarrow{\prec} \mathsf{HOD}_{\mathbb{R}}$ such that $j \restriction \mathrm{ORD} = \mathrm{id}$.*

Corollary 1.5. *Let* $\mathbb{F} = \mathtt{Add}(\omega, \lambda)$ *or* $\mathbb{F} = \mathtt{Random}_\lambda$ *where* $\lambda \geq \omega_1$. *Then in* $V^{\mathbb{F}}$, $\mathsf{HOD}_{\mathbb{R}} \models \neg\mathsf{AC}$ *and therefore no relation in* $\mathsf{HOD}_{\mathbb{R}}$ *defines a well-ordering of* $\mathbb{R}$. $\square$

Remark 1.6. If $\lambda < \omega_1$, then $\mathtt{Add}(\omega, \lambda) \cong \mathtt{Add}(\omega, 1)$ and $\mathtt{Random}_\lambda \cong \mathtt{Random}_\omega$. If G is generic over V for either of these forcings, there is a real r such that $V[G] = V[r]$. Thus, $V = L$ provides an example showing that $\lambda \geq \omega_1$ is necessary in the statement of the Corollary.

Proof of Lemma 1.4: Let G be $\mathbb{F}$-generic over V, where $\mathbb{F}$ is as in the statement of the lemma. By standard arguments (See Section 3.1 for the case $\mathbb{F} = \mathtt{Random}_\lambda$), $G \cong G_0 \times G_1$, where G_0 is $\mathbb{F}$-generic over V and G_1 is $\mathbb{F}^{V[G_0]}$-generic over $V[G_0]$. Let $\mathbb{R}_0 = \mathbb{R}^{V[G_0]}$ and $\mathbb{R}_1 = \mathbb{R}^{V[G]}$. In $V[G]$ we are to define a nontrivial $j : \mathsf{HOD}_{\mathbb{R}_0} \xrightarrow{\prec} \mathsf{HOD}_{\mathbb{R}_1}$ such that $j{\restriction}_{\mathrm{ORD}} = \mathrm{id}$.

For this, notice that any $x \in \mathsf{HOD}_{\mathbb{R}_i}$, $i = 0, 1$, has the form $\tau(\vec{r}, \vec{\alpha})$ where $\vec{r} \in \mathbb{R}_i$, $\alpha \in \mathrm{ORD}$, and τ is some term in the language of $\mathsf{HOD}_{\mathbb{R}}$.[1]

Define j by

$$j\big(\tau(\vec{r}, \vec{\alpha})^{\mathsf{HOD}_{\mathbb{R}_0}}\big) = \tau(\vec{r}, \vec{\alpha})^{\mathsf{HOD}_{\mathbb{R}_1}}.$$

We claim j works.

Let $\varphi(v_0, \ldots, v_n)$ be any formula, let $\tau_0(\vec{v}_0, \vec{v}_1), \ldots, \tau_n(\vec{v}_0, \vec{v}_1)$ be terms, and let $x_0, \ldots, x_n \in \mathsf{HOD}_{\mathbb{R}_0}$ be given by $x_i = \tau_i(\vec{r}_i, \vec{\alpha}_i)^{\mathsf{HOD}_{\mathbb{R}_0}}$. By composing each τ_i with some projections, we may assume $\vec{r}_i = \vec{r}$, $\vec{\alpha}_i = \vec{\alpha}$ for all i. Let $\psi(\vec{v}_0, \vec{v}_1) \equiv \varphi\big(\tau_0(\vec{v}_0, \vec{v}_1), \ldots, \tau_n(\vec{v}_0, \vec{v}_1)\big)$ and $\mu(\vec{v}_0, \vec{v}_1) \equiv \mathsf{HOD}_{\mathbb{R}} \models \psi(\vec{v}_0, \vec{v}_1)$.

Let $X \in \mathcal{P}_{\omega_1}(\lambda)$ be such that $r \in V[G_0{\restriction}_X]$, and let $\mathcal{G}_0$ and $\mathcal{G}_1$ be $\mathbb{F}^{V[G_0{\restriction}X]}$-generics over $V[G_0{\restriction}_X]$ such that $V[G_0{\restriction}_X][\mathcal{G}_0] = V[G_0]$ and $V[G_0{\restriction}_\beta][\mathcal{G}_1] = V[G]$.

Then

$$\begin{aligned}
\mathsf{HOD}_{\mathbb{R}_0} \models \psi(\vec{r}, \vec{\alpha}) &\iff V[G_0] \models \mu(\vec{r}, \vec{\alpha}) \\
&\iff \exists p \in \mathcal{G}_0 \, \big(V[G_0{\restriction}_X] \models p \Vdash_{\mathbb{F}} \mu(\check{\vec{r}}, \check{\vec{\alpha}})\big) \\
&\overset{(*)}{\iff} V[G_0{\restriction}_X] \models 1_{\mathbb{F}} \Vdash_{\mathbb{F}} \mu(\check{\vec{r}}, \check{\vec{\alpha}}) \\
&\overset{(*)}{\iff} \exists q \in \mathcal{G}_1 \, \big(V[G_0{\restriction}_X] \models q \Vdash_{\mathbb{F}} \mu(\check{\vec{r}}, \check{\vec{\alpha}})\big) \\
&\iff V[G] \models \mu(\vec{r}, \vec{\alpha}) \\
&\iff \mathsf{HOD}_{\mathbb{R}_1} \models \psi(\vec{r}, \vec{\alpha}),
\end{aligned}$$

[1]This language expands the language of set theory by closing under weak Skolem functions, see Definition 1.38 below.

where $(*)$ holds by the weak-homogeneity of $\mathbb{F}$.

This chain of equivalences implies immediately that j is well-defined and elementary. By definition, $j\restriction_{\mathrm{ORD}} = \mathrm{id}$, and we are done. $\square$

Remark 1.7. Notice that with the same notation as above, $j\restriction_{L(\mathbb{R}_0)} : L(\mathbb{R}_0) \xrightarrow{\prec} L(\mathbb{R}_1)$.

Essentially the same argument shows that if $\omega_1 \leq \lambda_1 \leq \lambda_2$, H_1 is $\mathtt{Random}_{\lambda_1}$- (respectively, $\mathtt{Add}(\omega,\lambda_1)$-) generic over V, and H_2 is $\mathtt{Random}_{\lambda_2}$- (respectively, $\mathtt{Add}(\omega,\lambda_2)$-) generic over $V[H_1]$, then in $V[H_1][H_2]$ there is a nontrivial embedding

$$j : L(\mathbb{R}^{V[H_1]}) \xrightarrow{\prec} L(\mathbb{R})$$

such that $j\restriction_{\mathrm{ORD}} = \mathrm{id}$. To see this, it suffices to argue that if $\varphi(x,y)$ is a formula, r is a real, α is an ordinal, and $\dot{\mathbb{R}}$ is a term (for the appropriate forcing) for the reals of the generic extension, then $1 \Vdash_{\mathtt{Random}_{\lambda_1}} L(\dot{\mathbb{R}}) \models \varphi(r,\alpha)$ iff $1 \Vdash_{\mathtt{Random}_{\lambda_2}} L(\dot{\mathbb{R}}) \models \varphi(r,\alpha)$.

Suppose $|\lambda_1| < |\lambda_2|$, and let $\mathbb{P}$ be the forcing for collapsing λ_2 to λ_1 with countable conditions, so $\mathbb{P}$ does not add any reals. If $\mathcal{G}$ is $\mathtt{Random}_{\lambda_1}$-generic over $V^{\mathbb{P}}$ then $\mathcal{G}$ is $\mathtt{Random}_{\lambda_1}$-generic over V, this is Fact 3.20. The same holds for $\mathtt{Random}_{\lambda_2}$-generic filters, and we are done by weak homogeneity. The argument for Cohen forcing is identical.

Thus, in general, all we can aspire to prove is the consistency of a definable well-ordering (within certain pointclass) assuming simultaneously some additional hypothesis which potentially interferes with this goal. There have been several partial results and obstructions in this direction, which we proceed to list along with the main results of this dissertation that continue this line of research.

1.1.1 Projective well-orderings

To begin with, we illustrate how a universe lacking a sufficiently rich large cardinal structure allows for pathological well-orderings:

Fact 1.8 (Gödel). *If $V = L$, then there is a Σ^1_2 well-ordering of the reals*[2]. $\square$

Since ZFC suffices to prove the Lebesgue measurability of analytic sets, no well-ordering of $\mathbb{R}$ can belong to a simpler pointclass than the one given by Gödel's result. As a matter of fact, this complexity characterizes the reals of L:

[2] A well-ordering W of $\mathbb{R}$ is a total relation, so $(x,y) \notin W$ iff $x = y$ or $(y,x) \in W$ for any x,y. Hence, for any reasonable pointclass Γ, a well-ordering is in Γ iff it is in $\neg\Gamma$. For example, a well-ordering of $\mathbb{R}$ is Σ^1_2 iff it is Δ^1_2.

Fact 1.9 (Mansfield). *Let $x \in \mathbb{R}$. There is a $\Sigma^1_2(x)$-well-ordering of $\mathbb{R}$ iff $\mathbb{R} = \mathbb{R}^{L[x]}$.* □

Mansfield's result (a nice proof of which is due to Kechris) can be seen as a strong argument against the existence of $\utilde{\Sigma}^1_2$-well-orderings. Pretty mild large cardinal hypotheses already justify our feeling; for example, if $\forall x \in \mathbb{R}\,(x^\sharp$ exists), then $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\,(y \notin L[x])$, and therefore there are no $\utilde{\Sigma}^1_2$-well-orderings (See also Fact 1.17, Theorem 1.13, and Fact 2.11.) In fact, the existence of sharps for reals gives us a stronger obstacle.

Theorem 1.10 (Martin). *If $\forall x \in \mathbb{R}\,(x^\sharp$ exists), then $\utilde{\Pi}^1_1$-determinacy holds and therefore all $\utilde{\Sigma}^1_2$-sets are Lebesgue measurable.* □

In contrast, much stronger large cardinal assumptions (although still quite weak within the large cardinal repertoire) are compatible with $\utilde{\Sigma}^1_3$-well-orderings.

Theorem 1.11 (Martin, Steel). *1. Suppose $L[\mathcal{E}]$ is a finestructural Martin-Steel model and*

$$L[\mathcal{E}] \models \texttt{There is no inner model with a Woodin cardinal.}$$

Then

$$L[\mathcal{E}] \models \texttt{There is a } \Sigma^1_3\texttt{-well-ordering of } \mathbb{R}.$$

In fact, the natural well-ordering of $L[\mathcal{E}]$, when restricted to $\mathbb{R}$, is of this complexity.

2. *Let M_n be the minimal (iterable) finestructural model for n Woodin cardinals. Then the reals of M_n admit a Σ^1_{n+3}-well-ordering. More precisely, there is in M_n a well-ordering W of $\mathbb{R}^{M_n}$ such that $M_n \models W$ `is` Σ^1_{n+2} and (if, say, there are n Woodin cardinals in V with a measurable above, or even if $M_n^\sharp$ exists) W is Σ^1_{n+3} in V.*

3. *If there are n Woodin cardinals with a measurable above, then all $\utilde{\Pi}^1_{n+1}$-sets are determined and therefore all $\utilde{\Sigma}^1_{n+2}$-sets are Lebesgue measurable. In particular, there are no $\utilde{\Sigma}^1_{n+2}$-well-orderings of the reals.*

4. *Hence, there are no projective well-orderings of the reals in the presence of ω Woodin cardinals.* □

This lands us within the $\utilde{\Sigma}^2_n$-realm very quickly:

Theorem 1.12 (Woodin). *If there are ω Woodins with a measurable above then $L(\mathbb{R}) \models$* AD *and in particular it is not a model of Choice.* □

Notice that Woodin's result is not a consistency result but an implication.

Before continuing to third-order pointclasses, let us step back momentarily and explore the projective hierarchy a bit longer.

Theorem 1.13 (Harrington?). *Suppose $\omega_1 = \omega_1^{L[x]}$ for some real x. Then there is a forcing extension of the universe with a $\underset{\sim}{\Sigma}^1_2$-well-ordering of $\mathbb{R}$.* □

In fact, it is not difficult to improve 1.13 to: There is a set generic extension of V with a $\underset{\sim}{\Sigma}^1_2$-well-ordering of $\mathbb{R}$ iff V is not closed under sharps, an assertion which is equivalent[3] to the failure of $\underset{\sim}{\Pi}^1_1$-determinacy in some $V^{\mathtt{Coll}(\omega,\kappa)}$.

Theorem 1.14 (Harrington). *There is a forcing extension of L where* MA *holds and there is a $\underset{\sim}{\Sigma}^1_3$-well-ordering of $\mathbb{R}$.* □

Thanks to arguments of Schindler, we can improve Harrington's result 1.13 in a provably optimal way:

Theorem 1.15. *Suppose that there is no inner model with ω strong cardinals. Then there is a set forcing extension of the universe with a projective well-ordering of the reals. In fact, K exists and if*

$$K \models \texttt{There are exactly } n \texttt{ strong cardinals},$$

then the well-ordering is $\underset{\sim}{\Delta}^1_{n+3}$. If $n = 0$ and V is not closed under sharps, the complexity can be improved to be $\underset{\sim}{\Delta}^1_2$.

That this is optimal follows from the following unpublished result due to Woodin:

Theorem 1.16 (Woodin). *Suppose there are n strong cardinals (and that V is closed under sharps, if $n = 0$.) Let λ be larger than all of them. Let G be $\mathtt{Coll}(\omega, <\lambda)$-generic over V. Then in $V[G]$ the Σ^1_{n+2} theory of the reals (with parameters from $\mathbb{R}^{V[G]}$) cannot be changed by further set forcing. In fact, the Σ^1_{n+2}-theory of the reals is Universally Baire[4], in particular measurable, and therefore no extension of $V[G]$ admits a $\underset{\sim}{\Sigma}^1_{n+2}$-well-ordering of $\mathbb{R}$.* □

See [St] Corollary 3.7 for a proof.

In a sense, Harrington's result 1.14 is optimal:

[3] By Theorem 1.10 and the results in Section 1.3.
[4] See Definition 3.73.

Fact 1.17 (Martin, Solovay). MA *implies the Lebesgue measurability of all* $\underset{\sim}{\Sigma}^1_2$*-sets.* □

We can improve his result in a different direction by obtaining a stronger forcing axiom:

Theorem 1.18. *Let* M *be a finestructural inner model with a strong cardinal and a measurable above. Suppose there are no inner models of* M *with Woodin cardinals. Then there is a forcing extension of* M *where the following hold*[5]*:*

- $\mathsf{SPFA}(\mathfrak{c}) + \mathsf{BSPFA}^{++}$.
- *Woodin's* ψ_{AC}.
- *There is a* Σ^1_5*-well-ordering of* $\mathbb{R}$.

This result cannot be improved much. For example, $\mathsf{MM}(\mathfrak{c})$ implies Projective Determinacy (See [W], Theorem 9.73) and therefore is incompatible with the existence of a projective well-ordering. Also, $\mathsf{SPFA}(\mathfrak{c}) + \mathsf{BSPFA}^{++}$ does not imply ψ_{AC}, by an easy modification of the argument in [AsWe]. The complexity of the well-ordering, however, is probably not optimal.

On the other hand, if we restrict our attention to forcing extensions of inner models, then some improvements are possible (nice behavior of simpler projective pointclasses can be imposed). For example:

Theorem 1.19 (Friedman, Schindler). *Let* $n > 0$ *and let* M^n *be the minimal iterable finestructural inner model with* n *strongs and an inaccessible above. Then there is a forcing extension of* M^n *in which all* $\underset{\sim}{\Sigma}^1_{n+3}$*-sets are Lebesgue measurable and there is a* Σ^1_{n+5}*-well-ordering of* $\mathbb{R}$. □

Notice that the well-ordering obtained is *lightface*. Allowing the presence of parameters, Friedman and Schindler prove a stronger result:

Theorem 1.20 (Friedman, Schindler). *Let* M_l^0 *be the minimal iterable finestructural inner model closed under the existence of sharps*[6]*, and for* $n > 0$ *let* M_l^n *be the minimal iterable finestructural inner model with* n *strongs. Then there is a forcing extension of* M_l^n *where the following hold:*

[5] See Definitions 2.17, 2.18 and 2.20 below.
[6] See Definition 1.38 below.

- *All* $\underset{\sim}{\Sigma}^1_{n+2}$*-sets are universally Baire.*
- *All* Σ^1_{n+3}*-sets are Lebesgue measurable and have the property of Baire.*
- *There is a* Π^1_{n+4}*-singleton* $a \in \mathbb{R}$.
- *There is a* $\Sigma^1_{n+3}(a)$*-well-ordering of* $\mathbb{R}$. $\square$

Our results on projective well-orderings are the content of Chapter 2.

1.1.2 Third-order definable well-orderings

Let us now step beyond the projective hierarchy and $L(\mathbb{R})$. The determinacy result 1.12 can be improved *under* CH. In fact, a remarkable generic absoluteness theorem can be obtained:

Theorem 1.21 (Woodin). *Suppose that either there is a proper class of measurable Woodin cardinals, or a proper class of strongly compact cardinals. Suppose that* CH *holds. Then all* $\underset{\sim}{\Sigma}^2_1$*-sets are determined and the* Σ^2_1*-theory of the reals (with real parameters from the ground model) is* ***generically invariant*** *with respect to extensions satisfying* CH. $\square$

More precisely, generic invariance of a class Γ of sentences with respect to a statement ϕ means that whenever $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration of set forcings such that $V^{\mathbb{P}} \models \phi + 1_{\dot{\mathbb{Q}}} \Vdash \phi$, then for all $\psi \in \Gamma$, $V^{\mathbb{P}} \models \psi$ iff $V^{\mathbb{P}*\dot{\mathbb{Q}}} \models \psi$. Since we are stating 1.21 under the assumption of CH in the ground model, generic invariance in this case can be restated as claiming that

$$V \equiv_{\underset{\sim}{\Sigma}^2_1} V^{\mathbb{P}}$$

whenever $V^{\mathbb{P}} \models$ CH, the boldface class intending to express that real parameters from the ground model are allowed.

That $\underset{\sim}{\Sigma}^2_1$ cannot be lifted to larger pointclasses without additional assumptions is the content of the following result of Abraham and Shelah which, in a sense, was the inspiring force behind this thesis.

Theorem 1.22 (Abraham, Shelah [ASh]). *There is a poset* $\mathbb{P}$ *of size* $2^{2^{\aleph_1}}$ *which adds no new reals and such that, in* $V^{\mathbb{P}}$, CH *holds and the reals admit a* Σ^2_2*-well-ordering. In fact, any given relation of the reals may be encoded by a variant of* $\mathbb{P}$ *in a* Σ^2_2*-way in some generic extension.* $\square$

The encoding in the Abraham-Shelah result is obtained by imposing strong restrictions on the Aronszajn trees of the resulting extension in such a way that some products are special, while others are Suslin trees.

It must be mentioned that this restriction implies that $\diamondsuit$ fails hopelessly in the model of Theorem 1.22.

On the other hand, if CH fails then an improvement is possible:

Theorem 1.23 (Woodin). *Suppose there is a weakly compact cardinal κ. Then there is a forcing extension that preserves cofinalities where $\mathfrak{c} = \kappa$, MA(σ-centered) holds, and there is a Σ^2_1-well-ordering of $\mathbb{R}$.* $\square$

Theorem 1.24 (Abraham, Shelah [ASh1]). *Let κ be the first inaccessible*[7]*. Then there is a cofinality preserving forcing extension where $\mathfrak{c} = \kappa$, MA holds, and there is a Σ^2_1-well-ordering of $\mathbb{R}$.* $\square$

At the cost of a small value of $\mathfrak{c}$ the large cardinals are superfluous:

Theorem 1.25 (Solovay). *There is a forcing extension where $\mathfrak{c} = \aleph_2$, MA(σ-centered) holds, and there is a Σ^2_1-well-ordering of $\mathbb{R}$.* $\square$

And even better:

Theorem 1.26 (Abraham, Shelah [ASh2]). *Solovay's result 1.25 holds with MA(σ-centered) strengthened to MA.* $\square$

Another instance of this theme constitutes the main result of this dissertation:

Fact 1.27. *Assume $\mathfrak{c}$ is real-valued measurable*[8]*. Then*

1. *MA_{ω_1}(σ-centered) and OCA fail.*
2. *There is no well-ordering of $\mathbb{R}$ in $L(\mathbb{R})$.*

The fact follows from a characterization of real-valued measurability in terms of elementary embeddings due to Solovay, see 3.6, 3.31, 3.41, and 3.49 below.

[7] The published version of this result claims that κ can be any inaccessible, but the argument given there rests on a variant of Easton support iterations the combinatorics of which require κ to be the first inaccessible.

[8] See Definition 3.1.

Theorem 1.28. *Suppose that κ is measurable and $2^\kappa = \kappa^+$. Then there is a forcing notion $\mathbb{F}$ of size κ that makes $\kappa = \mathfrak{c}$ while preserving its real-valued measurability, and adding a Δ^2_2-well-ordering of the reals.*

If the universe is thin then the complexity of the well-ordering can be simplified:

Theorem 1.29 (Woodin). *If $V = L[\mu]$, the minimal model for a measurable, then there is an extension where $\mathfrak{c}$ is real-valued measurable and there is a Σ^2_1-well-ordering of $\mathbb{R}$.*

In fact, at least if we grant the existence of enough large cardinals, this result can be improved significantly:

Theorem 1.30 (Woodin). *Assume there is a proper class of Woodin cardinals. Then it is Ω-consistent*[9] *that $\mathfrak{c}$ is real-valued measurable and there is a $\underset{\sim}{\Sigma}^2_1$-well-ordering of $\mathbb{R}$.* □

In Chapter 3 we sketch the proof of 1.29. Theorem 1.30 is still unpublished; it builds on the results of [W], and makes use of an appropriate version of the following Lemma[10]. Stevo Todorčević (in personal communication) has indicated that 1.31 was known to James Hirschorn who (in personal communication) attributed it to an unpublished paper by Baumgartner. To the best of my knowledge no proof of Lemma 1.31 had appeared in print prior to this thesis. To ease the presentation, we do not state optimal hypotheses. Compare with [GoSh], Section 3.

Lemma 1.31. *Suppose* $\mathsf{PFA}(\mathfrak{c})$ *holds. Then, after adding $\aleph_1$ random reals (i.e, forcing with* $\mathtt{Random}_{\aleph_1}$*) all trees of size ω_1 are sealed in the sense that if an outer model M sees a cofinal branch b through a tree $\mathcal{T} \in V$ such that b is not already in V, then $\omega_1^V < \omega_1^M$.*

Proof: First, Baumgartner has an argument showing that if a tree of size and height ω_1 has at most ω_1 many branches, and all trees of size and height ω_1 with no branches are special, then the result holds. This he achieves by generalizing the notion of special to these trees. Given an ω_1-enumeration of the branches through the tree, Baumgartner defines a subtree with no cofinal branches, thus special, and shows how to extend the specializing function to the whole tree. See [Ba].

An argument of Laver shows that if MA holds, after adding random reals all trees of size ω_1 with no branches are special. See [BJ] for a proof.

[9] See Definition 3.94.

[10] However, see the last footnote on Chapter 3.

Suppose $\neg$wKH, MA and MA(σ-closed) hold. $\neg$wKH is the assertion that no trees of size ω_1 have more than ω_1 many branches.

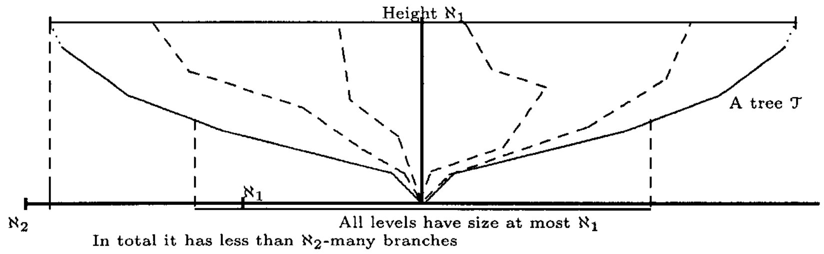

Figure 1.2: $\mathcal{T}$ is not a weak Kurepa tree

Let $\mathbb{Q} = \texttt{Random}_{\aleph_1}$ and let τ be (with Boolean value 1) a $\mathbb{Q}$-name for a tree of size ω_1. Let (in V) $\mathbb{P}$ be the collapse of $\mathfrak{c}$ to ω_1. We argue that $\neg$wKH is preserved by $\mathbb{Q}$. This we do by showing that forcing by $\mathbb{P}$ does not add a $\mathbb{Q}$-name for a branch b through τ not in $V^{\mathbb{Q}}$.

If this holds, then the set of branches of τ in $(V^{\mathbb{P}})^{\mathbb{Q}}$ is the same as the set of branches of τ in $V^{\mathbb{Q}}$. Since $(V^{\mathbb{P}})^{\mathbb{Q}} \models$ CH, τ has only ω_1 many branches in that extension. There is therefore a $\mathbb{Q}$-term ς for a subtree of τ with no cofinal branches, as defined by Baumgartner. This term must be in V, by MA(σ-closed).

By Laver's result, $V^{\mathbb{Q}} \models \varsigma$ `is special`. Baumgartner's argument shows then that $V^{\mathbb{Q}} \models \tau$ `is special`, and the proof is complete.

So only the fact that $\mathbb{P}$ does not add a $\mathbb{Q}$-name b for a "new" branch needs arguing. This is proven by a standard elementary substructure argument. Suppose otherwise. Without loss, this holds with boolean value 1, i.e.,

$$1 \Vdash_{\mathbb{P}} [\ b \texttt{ is a } \mathbb{Q}\texttt{-name, } b \texttt{ is not in } \check{V}\texttt{, and}$$
$$1 \Vdash_{\mathbb{Q}} [b \texttt{ is a branch of } \tau\texttt{.}]\ \ ]$$

Let η be regular, large, and look at a suitable countable $X \prec H_\eta$.

Let G be $\mathbb{Q}$-generic. Then $X[G] \prec H_\eta[G]$. Let N be the collapse of X and $N[g]$ the collapse of $X[G]$. Notice $g = G\lceil_{\omega_1^X}$. In $N[g]$ there is a tree T which interprets τ and is part of the actual tree τ_G in $V[G]$.

Since b is "new", in V there is a perfect tree of conditions g in $\mathbb{P}$, all of which are $\mathbb{P}$-generic over N and force incompatible requirements on b.

By refining these conditions, we can assume they decide incompatible initial segments b^g of b. All these segments are in V. Since H induces a generic h over $N[g]$—because $N[g]$ is in V—, b^g is contained in τ_H, the interpretation of the name τ in $V[H]$.

If G_g is $\mathbb{P}$-generic and extends one of these conditions g, then b_G is a $\mathbb{Q}$-name for a branch of τ, and if H is Q-generic over $V[G]$, then b_G extends b^g. There are $2^{\aleph_0}$ many incompatible b^g. This contradicts $\neg\mathsf{wKH}$ in V, and we are done. $\square$

In Chapter 3 we also introduce a strengthening of real-valued measurability, real-valued hugeness, which implies the existence of many real-valued measurable cardinals and in consistency strength goes even further[11]. It is consistent that $\mathfrak{c}$ is real-valued huge.

Theorem 1.32 (Woodin). *If $\mathfrak{c}$ is real-valued huge, then there are no third-order definable well-orderings of the reals. In fact, there is an uncountable λ such that V and $V^{\mathtt{Random}_\lambda}$ are $\underset{\sim}{\Sigma}^2_\omega$-elementary equivalent.*

These results constitute the main body of the thesis, Chapter 3.

We close this Section by mentioning a recent result due to Woodin, which in the appropriate context provides a generic absoluteness theorem for a stronger pointclass than $\underset{\sim}{\Sigma}^2_1$. Notice CH is a Σ^2_1 statement, and $\diamondsuit$ is Σ^2_2. An important difference between this result and Theorem 1.21 is that the determinacy of the required pointclasses is not known to follow in the presence of $\diamondsuit_{gen}$ from any instances of large cardinal axioms. See [W2] for further details.

Definition 1.33. Let $\varphi(x_1,\dots,x_n)$ be a formula and let $A \subseteq \mathbb{R}$. The *A-Neeman game* given by φ is as follows:

Two players I and II alternate playing digits $a_\alpha \in 2$, for ω_1 many moves, to form a set $a \in 2^{\omega_1}$.

I wins iff there is a closed unbounded set $C \subseteq \omega_1$ such that for all $\alpha_1 < \cdots < \alpha_n$ in C,

$$(H_{\omega_1}, \in, a, A) \models \varphi(\alpha_1,\dots,\alpha_n).$$

Definition 1.34 (Generic Diamond). We say that *generic diamond* holds, $\diamondsuit_{gen}$, iff

$$H_{\omega_2} \equiv_{\Sigma_2} H_{\omega_2}^{V^{\mathrm{Coll}(\omega_1,\mathfrak{c})}}.$$

[11]How much we have not explored yet (It goes well beyond the existence of Woodin cardinals. For example, $\mathbb{R}^\sharp$ exists and $\mathsf{AD}^{L(\mathbb{R}^\sharp)}$ holds.) However, it seems quite reasonable to expect that it is much stronger than anything which can be attained with current finestructural or descriptive set-theoretic techniques.

Generic Diamond was defined by Woodin. It is consistent, being true in $V^{\mathrm{Coll}(\omega_1,\mathfrak{c})}$. Notice that $\diamondsuit$ itself is a trivial consequence of $\diamondsuit_{gen}$.

Theorem 1.35 (Woodin). *Suppose there is a proper class of supercompact cardinals. Let Γ^∞ be the collection of all universally Baire subsets of $\mathbb{R}$. Suppose for each $A \in \Gamma^\infty$,*

$$\mathsf{ZFC} + \diamondsuit_{gen} \vdash_\Omega \texttt{All } A\texttt{-Neeman Games are determined.}$$

Then for all $A \in \Gamma^\infty$ and all Σ^2_2-formulas $\varphi(x)$, either $\mathsf{ZFC}+\diamondsuit_{gen} \vdash_\Omega \varphi(A)$ or $\mathsf{ZFC}+\diamondsuit_{gen} \vdash_\Omega \neg\varphi(A)$. $\square$

The pointclass Γ^∞ is formally introduced in Definition 3.73 and the surrounding discussion. For the definition of Ω-logic, see Section 3.6. If the Ω-conjecture holds then Theorem 1.35 implies that in the presence of large cardinals, and under the appropriate determinacy assumptions, then Σ^2_2-generic invariance holds with respect to extensions satisfying $\diamondsuit_{gen}$, in the sense explained after the statement of Theorem 1.21. In particular this seems to impose a serious obstacle to the project of improving the Abraham-Shelah Theorem 1.22 by strengthening CH to $\diamondsuit$.

Even if the stated determinacy hypothesis fails, it is still believed that the $\utilde{\Sigma}^2_2$-theory of the reals can be made conditionally generically invariant, the conditional being $\diamondsuit_{gen}$ or maybe even $\diamondsuit$.

Conjecture 1.36 (Woodin). *Suppose that there is a proper class of supercompact cardinals. Suppose that $\diamondsuit_{gen}$ holds. Then the Σ^2_2-theory of the reals (with real parameters from the ground model) is generically invariant with respect to extensions satisfying $\diamondsuit_{gen}$.*

1.2 Forcing Axioms

Woodin's result 1.29 makes essential use of the fact that the ground model is "small" and thus can be identified in the extension in a projective way. This gives us some leeway and, as mentioned in Section 3.6, if the Ω-conjecture holds then under appropriate large cardinal hypotheses, Theorem 1.28 can be improved by replacing Σ^2_2 with Σ^2_1. This improvement of Woodin is due (assuming the Ω-conjecture) to a "covering" lemma, a finestructural analysis of models of AD^+,[12] and even to Lemma 1.31, which builds on the fact that appropriate forcing axioms rule out the existence of weak Kurepa trees.

[12]See Section 1.4 for the definition of AD^+.

The last chapter of the thesis provides a new twist to the well-known open question of whether there is a forcing extension of V that collapses $\aleph_{\omega+1}$ to $\aleph_2$. We show:

Lemma 1.37. *Suppose M is an inner model of* GCH.

- *Suppose* PFA($\mathfrak{c}$) *holds. Then, without loss of generality, $\omega_1^M = \omega_1$. More precisely, there is an inner model N, $M \subseteq N$, such that $N \models$* GCH, *$\omega_1 = \omega_1^N$, and ω_2^V is inaccessible in N iff it is inaccessible in M.*
- *Moreover, if* PFA($\mathfrak{c}$) *holds and ω_2^V is a successor cardinal in M, then $\omega_2^V = (\lambda^+)^M$, where $\mathrm{cf}(\lambda) = \omega$, and $\square^*_\lambda$ fails in M.*
- *Whenever M is an inner model of* GCH *correctly computing $\aleph_1$ and such that $\aleph_2^V = (\lambda^+)^M$, where $\mathrm{cf}^V(\lambda) = \omega$, then*
  - *In M the approachability property fails at λ and there are no uniformly almost disjoint sequences for λ, in particular $\mathrm{cf}^M(\lambda) = \omega$.*
  - *V is not a weakly proper forcing extension of M, and no inner model of V that computes ω_2 correctly satisfies* CH. *In particular, if* PFA($\mathfrak{c}$) *holds then there is a real r such that $M[r] \models \neg$*CH.
  - *Furthermore, if $\omega_2^V = \aleph_{\omega+1}^M$, then $\mathsf{VWS}_{\aleph_\omega}$ fails in M, and $(S^{\lambda^+}_{\omega_1})^M =_{\mathrm{NS}_{\omega_2}} S^{\omega_2}_{\omega_1}$.*

(See Chapter 4 for all unexplained notation. $A =_{\mathrm{NS}_{\omega_2}} B$ means that there is a club C subset of ω_2 such that $A \cap C = B \cap C$.)

1.3 A Review of Sharps

This Section is included because all references on sharps of general sets (not necessarily sets of ordinals) seem unsatisfactory one way or another, and this list of facts may prove useful as a guide to the reader here and elsewhere. Only a knowledge of the theory of $0^\sharp$ is required.

Definition 1.38. Let Y be a transitive set.

1. A *class of indiscernibles* for $L(Y), Y$ (informally, for $L(Y)$) is a class $I \subseteq \mathrm{ORD}$ such that for all $\vec{a}$ elements of Y and all $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ elements of I,

for any $\varphi(\vec{x}, y_1, \ldots, y_n)$ in the language of set theory,

$$L(Y) \models \varphi(\vec{a}, \vec{\alpha}) \iff L(Y) \models \varphi(\vec{a}, \vec{\beta}).$$ [13]

2. Let $\varphi(t, x_1, \ldots, x_n)$ be a formula in the language of set theory, expanded with constant symbols for Y and the elements of Y. A *weak Skolem function* for φ (with respect to $L(Y), Y$) is the function $f_\varphi : {}^nL(Y) \to L(Y)$ given by

$$f_\varphi(\vec{x}) = \begin{cases} y & \text{if } L(Y) \models y \texttt{ is the unique } z \texttt{ such that } \varphi(z, \vec{x}); \\ \emptyset & \text{if there is no such unique } z. \end{cases}$$

3. Let $Y \subseteq Z \subseteq L(Y)$. By $\mathcal{H}(L(Y), Z)$ we mean the closure of Z under weak Skolem functions.

4. Let I be a class of indiscernibles for $L(Y), Y$. We say that I *generates* $L(Y)$ iff

$$\mathcal{H}(L(Y), I \cup Y) = L(Y).$$

5. We say that $Y^\sharp$ *exists* iff there is a club proper class I of indiscernibles for $L(Y), Y$ such that $I \cup Y$ generates $L(Y)$ and, moreover, for any uncountable η such that $Y \in H_\eta$, $\mathcal{H}(L(Y), (I \cap \eta) \cup Y) = L_\eta(Y)$.

6. We say that $X^\sharp$ exists iff $Y^\sharp$ exists, where $Y = \text{TrCl}(X)$.

Fact 1.39. *If $X \in H_\eta$ and η is Ramsey, then $X^\sharp$ exists.* □

The assertion "$X^\sharp$ exists" refers to the existence of a proper class object. Solovay's realization (see [Sol]) is that just as in the case of sharps for reals, this is in fact equivalent to the existence of a set, and it is this set what we now call $X^\sharp$.

Definition 1.40. Let Y be transitive.

1. Let $\mathcal{L}_Y$ denote the language of set theory augmented with constants for the elements of $Y \cup \{Y\}$, and with ω many other constants c_n, $n \in \omega$, (to represent the first ω indiscernibles), and closed under terms for weak Skolem functions for formulas in the language of set theory.

[13] We consider the language of $L(Y)$ to be expanded by constants P_a for each $a \in Y \cup \{Y\}$. The natural interpretation of P_a is, of course, the set a.

2. An *EM blueprint* for Y (EM stands for Ehrenfeucht-Mostowski) is the theory in $\mathcal{L}_Y$ of some structure $(L_\eta(Y), \in, P_a, i_n)_{\substack{a \in Y \cup \{Y\} \\ n \in \omega}}$ where $Y \in H_\eta$ or $\eta = \text{ORD}$, and $\langle i_n : n < \omega \rangle$ is the increasing enumeration of a set of indiscernibles for
$$(L_\eta(Y), \in, P_a)_{a \in Y \cup \{Y\}}.$$

3. Let Σ be an EM blueprint for Y, and let α be an ordinal. By $\Gamma(\Sigma, \alpha)$ we mean, provided that it exists and is unique (up to isomorphism), a model $\mathcal{M}_\alpha$ such that
   (a) $\mathcal{M}_\alpha \models \Sigma^*$, the restriction of Σ to the language $\mathcal{L}'_Y$ without constants for the indiscernibles.
   (b) There is a set $I \subset \text{ORD}^{\mathcal{M}_\alpha}$ such that $(I, \in^{\mathcal{M}_\alpha}) \cong (\alpha, \in)$ which is a set of indiscernibles for $\mathcal{M}_\alpha$.
   (c) $\mathcal{H}(\mathcal{M}_\alpha, I \cup \{ P_a^{\mathcal{M}_\alpha} : a \in Y \cup \{Y\} \}) = \mathcal{M}_\alpha$.

4. A set of sentences $\Sigma \subseteq \mathcal{L}_Y$ is a *remarkable character* for Y iff
   (a) Σ is an EM blueprint for Y. In fact, Σ extends "$\mathsf{ZF} + V = L(Y)$".
   (b) $\Gamma(\Sigma, \alpha)$ exists and is well-founded for all α.
   (c) For any term $t(x_0, \dots, x_{n-1})$ in $\mathcal{L}_Y$, the sentence
   $$\text{“}t(c_0, \dots, c_{n-1}) \in \text{ORD} \longrightarrow t(c_0, \dots, c_{n-1}) < c_n\text{”}$$
   belongs to Σ.
   (d) For any term $t(x_0, \dots, x_{m+n})$ in $\mathcal{L}_Y$, the sentence
   $$\text{“}t(c_0, \dots, c_{m+n}) < c_m \longrightarrow t(c_0, \dots, c_{m+n}) = t(c_0, \dots, c_{m-1}, c_{m+n+1}, \dots, c_{m+2n+1})\text{”}$$
   belongs to Σ.
   (e) Σ satisfies the *witness condition*:
   Whenever $\exists x\, \varphi(x) \in \Sigma$, there is a term t all of whose constants for indiscernibles already appear on $\varphi(x)$, and such that $\varphi(t) \in \Sigma$.

The witness condition is the key condition that remarkable characters for reals (or more generally for sets of ordinals) satisfy automatically, because Skolem terms are definable in $L[x]$, $x \in \mathbb{R}$, since $L[x]$ has a definable well-ordering. Its importance lies in that it allows us to prove the following basic fact:

Lemma 1.41 (Solovay). *If Σ is a remarkable character for a transitive set Y, then*

1. *For all α, the sequence I^α of indiscernibles of $\Gamma(\Sigma,\alpha)$ with $(I^\alpha, \in^{\Gamma(\Sigma,\alpha)}) \cong (\alpha, \in)$ satisfies that for any formula $\varphi(x_1,\dots,x_n)$ in the language $\mathcal{L}'_Y$, $\varphi(c_1,\dots,c_n) \in \Sigma$ iff there is a $\in^{\Gamma(\Sigma,\alpha)}$-increasing sequence $a_1,\dots,a_n$ of elements of I^α such that $\Gamma(\Sigma,\alpha) \models \varphi(a_1,\dots,a_n)$.*

2. *For any cardinal η such that $Y \in H_\eta$,*
$$\Gamma(\Sigma,\eta) \cong L_\eta(Y).$$

3. *For all α, I^α is closed unbounded in $\mathrm{ORD}^{\Gamma(\Sigma,\alpha)}$.*

4. *If $\alpha < \beta$, then I^β end-extends I^α (seen as subsets of $\mathrm{ORD}^{L_\eta(Y)}$ for any cardinal η such that $\beta, Y \in H_\eta$.)*

5. *For any η such that $Y \in H_\eta$, $\mathcal{H}(L(Y), I^\eta \cup Y) = L_\eta(Y) \prec \mathcal{H}(L(Y), \bigcup_\alpha I^\alpha \cup Y) = L(Y)$.*

6. *Let Σ' be any remarkable character for Y. Then $\Sigma' = \Sigma$.* □

Corollary 1.42 (Solovay). *Let Y be transitive. Then $Y^\sharp$ exists iff there is a remarkable character for Y.* □

Remark 1.43. In truth, Solovay only argued these results for sharps of sets of reals (or, more precisely, for $\mathbb{R}^\sharp$), but the arguments for $0^\sharp$ really lift straightforwardly.

It follows that it makes sense to define sharps in terms of the remarkable characters whose existence they guarantee:

Definition 1.44. Let X be a set and let Y be its transitive closure. Then $X^\sharp := \Sigma$, for Σ the unique remarkable character for Y.

See [So1], where the general notion of sharps is introduced, in the context of subsets of reals.

Notice the definition of $Y^\sharp$ is absolute in the sense that if $W \supseteq V$ is an outer model and $Y^\sharp \in V$, then
$$W \models (Y^\sharp)^V = Y^\sharp.$$

The following is ancient, but I have been unable to find a reference:

Fact 1.45. *Let $\mathbb{P}$ be a poset, and suppose $x^\sharp \in V^{\mathbb{P}}$, where x is a real coding a set $X \in V$. Then $X^\sharp \in V$.* □

It follows from the fact that Jensen's covering lemma relativizes to all sharps, so $L[X]$ satisfies covering above η, where $X \in H_\eta$, iff $X^\sharp$ does not exist. Since set sized forcing preserves a tail of the class of cardinals, if $\mathbb{P}$ is a poset and $X^\sharp$ exists in $V^{\mathbb{P}}$, then $X^\sharp$ exists in V.

Fact 1.46 (Solovay). *If $X^\sharp$ exists, then the truth sets of $L(X)$ and $L[X]$ are definable.* □

The following example must be folklore, it was shown to me by Woodin. It illustrates that we cannot make do in the definition of $X^\sharp$ without the witness condition:

Recall first that after adding ω_1 Cohen reals, no well-ordering of $\mathbb{R}$ belongs to $L(\mathbb{R})$. This follows immediately from Lemma 1.4.

Claim 1.47. *Let $V = L[\mu]$,*[14] *and let G be $\mathtt{Add}(\omega, \omega_1)$-generic over V. Then*

1. *$(\mathbb{R}^\sharp)^{V[G]}$ exists.*
2. *$(\mathbb{R}^\sharp)^{V[G]} \cap V \in V$.*
3. *$(\mathbb{R}^\sharp)^{V[G]} \cap V$ satisfies conditions 4.(a)–(d) of Definition 1.40 for $(\mathbb{R}^\sharp)^V$.* □

If we could dispense with the witness condition in Definition 1.40, it would follow from the claim that $\mathbb{R}^{L[\mu]}$ is not well-orderable by a well-ordering in $L(\mathbb{R})^{L[\mu]}$. This is absurd, since in fact $\mathbb{R}^{L[\mu]}$ admits a Δ^1_3-well-ordering.

Remark 1.48. Of course, the same arguments generalize to larger sharp-like objects, like daggers or pistols.

The theory of sharps is usually recalled in connection with finestructural arguments. In this context, $X^\sharp$ is usually defined as a particular kind of mouse.

Fact 1.49. *Let X be a set. Then $X^\sharp$ exists iff there is an active X-mouse.* □

There is therefore no lack of generality in using this approach. We actually obtain quite more information than what was stated in Fact 1.49. For example, by standard

[14] See Definition 3.67 below.

techniques a mouse as in 1.49 is unique if it exists, and so we can identify it with $X^\sharp$. Moreover, for example if $x \in \mathbb{R}$, $x^\sharp$ and the minimal active x-mouse share the same Turing degree.

1.4 A quick overview of AD^+

We have occasion to mention AD^+ in a couple of places. This brief Section intends to provide the relevant definitions. For a (very brief, but more complete) introduction to AD^+, see [W], Section 9.1. AD^+ is a technical strengthening of AD, due to Hugh Woodin.

It intends to axiomatize those sentences φ such that whenever $M \subseteq N$ are transitive models of $\mathsf{ZF}^{-\varepsilon} + \mathsf{AD}$ with the same reals and such that every set of reals in M is Suslin[15] in N, then $M \models \varphi$.

The assumption on "external Susliness" of the sets of reals in M guarantees that they possess (descriptive set-theoretic) scales (in N). The intuition is that many consequences of AD depend on the existence of scales for different sets of reals, but an examination of the arguments tends to show that the scales themselves need not be in the model. This is why AD^+ was originally known as "AD within scales", and some authors (most notably, Steve Jackson) still refer to it in this way.

The usual motivation for AD^+ is less technical: AD^+ intends to lift to models $M = L(\mathcal{P}(\mathbb{R}))^M$ the rich theory that $L(\mathbb{R})$ satisfies under the assumption of $\mathsf{AD}^{L(\mathbb{R})}$.

Definition 1.50 (Woodin). Assume ZF. AD^+ is the following theory:

1. $\mathsf{DC}_{\mathbb{R}}$.

2. Every set of reals is ∞-*Borel*. This is to say: For all $A \subseteq \mathbb{R}$ there is an ordinal α, a set of ordinals S, and a formula $\phi(x, y)$ such that

$$A = \{\, r \in \mathbb{R} : L_\alpha[S, r] \models \phi(S, r)\,\}.$$

3. Suppose $\lambda < \Theta$, where $\Theta := \sup\{\,\alpha : \exists f : \mathbb{R} \to \alpha\,(f \text{ is onto})\,\}$. Endow λ with the discrete topology, and $\lambda^\omega := {}^\omega\lambda$ with the product topology. Assume $\pi : \lambda^\omega \to \omega^\omega$ is continuous. Then for each $A \subseteq \mathbb{R}$, the set $\pi^{-1}``A$ is determined.

[15] A set $A \subseteq \mathbb{R}$ is Suslin iff it is κ-Suslin for some cardinal κ, which is to say that there is a tree T on $\omega \times \kappa$ whose projection is A. The cardinals κ for which there is such a set A which in addition is not α-Suslin for any $\alpha < \kappa$ are called *Suslin* cardinals.

Thus, AD^+ strengthens AD. It is usually not required that $\mathsf{DC}_{\mathbb{R}}$ be part of AD^+, but the theory of AD^+ is always studied within the context of $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$. Moreover, the most important open question in the theory of determinacy, Question 1.51 below, is usually understood as the conjunction of three questions, one for each requirement in Definition 1.50, *even* if the base theory in the last two questions is assumed to be $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$.

Question 1.51 (Woodin). *Assume* ZF. *Are* AD *and* AD^+ *equivalent?*

It is not immediate even that $L(\mathbb{R}) \models \mathsf{AD} \to \mathsf{AD}^+$. This is the content of the following results:

Theorem 1.52 (Kechris [Ke]). *Assume* $V = L(\mathbb{R}) \models \mathsf{ZF}+\mathsf{AD}$. *Then* $\mathsf{DC}_{\mathbb{R}}$ *(and therefore* DC*) holds.* □

Kechris's result is seminal in that it constitutes what can probably be construed as the first example of (a very early form of) the core model induction.

Theorem 1.53. *Assume* $V = L(\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD}$. *Let* $\lambda < \Theta$ *be endowed with the discreet topology. Suppose* $A \subseteq \omega^\omega$, $n < \omega$, *and* $F : (\omega^\omega)^n \times \lambda^\omega \to \omega^\omega$ *is continuous. Consider the game* $\mathsf{G}_{\lambda,F,A}$ *on* λ *where I, II play* $\alpha_0, \alpha_1, \dots$ *producing* $\vec{\alpha} \in \lambda^\omega$, *and I wins iff*

$$\exists x_1\, \forall x_2 \dots Q x_n\, (F(x_1, x_2, \dots, x_n, \vec{\alpha}) \in A),$$

where Q *is either* $\exists$ *if* n *is odd, or* $\forall$ *if* n *is even (and bigger than* 0*). Then* $\mathsf{G}_{\lambda,F,A}$ *is determined.* □

This is a consequence of an earlier version due to Moschovakis, and of Solovay's Basis Theorem. I do not know who was first to prove it. As stated, it appears as Theorem 2.17 in [Ja].

Theorem 1.54 (Woodin). *Assume* $\mathsf{ZF} + \mathsf{DC}$. *Let* μ *be a fine measure on* $\mathcal{P}_{\omega_1}(\mathbb{R})$. *Then* $A \subseteq \mathbb{R}$ *is* ∞*-Borel iff for some set* $S \subseteq \mathrm{ORD}$, $A \in L(S, \mathbb{R})$. □

I think this result is still unpublished. The proof uses a Prikry-like iteration of Vopenka-like forcings. Under $\mathsf{AD} + \mathsf{DC}$ the Martin measure generates a fine measure as required, and therefore we have:

Corollary 1.55. $L(\mathbb{R}) \models \mathsf{AD} \to \mathsf{AD}^+$. □

That the given axiomatization of AD^+ has indeed succeeded in capturing the intuitive notion it intended to is the content of the following (folklore?) result:

Theorem 1.56. *Let $M \subseteq N$ be transitive models of $\mathsf{ZF}^{-\varepsilon} + \mathsf{AD}$ such that $\mathbb{R}^M = \mathbb{R}^N$ and every set of reals in M is Suslin in N. Then $M \models \mathsf{AD}^+$.* $\square$

The following is not really an AD^+-result, but AD^+ builds on analogues of it for models larger than $L(\mathbb{R})$:

Theorem 1.57. $L(\mathbb{R}) \models \exists S \subseteq \Theta\,(\mathsf{HOD} = L[S])$. $\square$

The set S as in 1.57 is obtained by a version of Vopenka's forcing due to Woodin that can add $\mathbb{R}$ to $\mathsf{HOD}^{L(\mathbb{R})}$, see [HaMaW] for a proof. Variants of this forcing are very useful at different points during the development of the AD^+ theory.

1.5 Notation

We have tried to use only standard set-theoretic notation, as in [Ku] or [J].

Our base theory is ZFC, usually augmented with large cardinals, but if it makes for cleaner statements or proofs we do not hesitate to adopt the language of proper classes. All these uses can be dispensed with at the expense of clumsier (first-order) renderings of the propositions being studied.

As usual "iff" abbreviates "if and only if". The end of a proof is indicated by an empty box, $\square$, which we also include at the end of statements whose proof is omitted. The end of a proof-within-a proof is indicated by $\triangle$, and the end of proofs nested within this level is marked by $\triangledown$.

If f is a function, $f \vdots x \to y$ means that $\mathrm{dom}\,(f) \subseteq x$ (and $f``(x) := \{\, f(z) : z \in x \,\} \subseteq y$).

Let f be a function and let $X \subseteq \mathrm{dom}\,(f)$. By $x \in X \mapsto f(x)$ or $\lambda x \in X.f(x)$ we mean exactly the same as by $f{\restriction}_X$. If X is clear from context we might omit it, thus writing $x \mapsto f(x)$ or $\lambda x.f(x)$.

For forcing notions we follow the western tradition of writing $p \leq q$ to indicate that the condition p is *stronger* than the condition q.

By forcing we mean set forcing, i.e., forcing by a set sized partial order (a poset). We will emphasize the word "set" occasionally, but no class forcing will be used at all.

The main reason is that set forcing and proper class forcing should be considered as having different ontological status. While set forcing is always small for large cardinals sufficiently high up in the hierarchy, and therefore essentially preserves the large cardinal structure of the universe, class forcing may destroy it completely. For example, recall Jensen's celebrated coding theorem that there is a class forcing $\mathbb{P}$ so that the extension $V^{\mathbb{P}}$ satisfies $V^{\mathbb{P}} = L[r]$ for some real r, and therefore $V^{\mathbb{P}}$ admits a projective (even $\utilde{\Sigma}^1_2$!) well-ordering of the reals, no cardinal is strong in $V^{\mathbb{P}}$, etc.

Following Woodin's usage, by $\mathtt{Add}(\kappa, \lambda)$ we mean the forcing that adds λ many Cohen subsets of κ, conditions are partial functions $p : \kappa \times \lambda \to 2$ with $|p| < \kappa$, ordered by $p \leq q$ if and only if $p \supseteq q$. In Kunen's notation this is $\mathsf{Fn}(\kappa \times \lambda, 2, \kappa)$. A Cohen real is an $\mathtt{Add}(\omega, 1)$-generic real.

By $\mathtt{Coll}(\kappa, X)$ we mean the forcing that collapses X to size κ, where conditions are partial functions $p : \kappa \to X$ with $|p| < \kappa$ ordered by extension. In Kunen's notation this is $\mathsf{Fn}(\kappa, X, \kappa)$. The Lévy collapse of ordinals smaller than λ to size κ is $\mathtt{Coll}(\kappa, < \lambda)$.

If $\mathbb{P}$ is a poset, by $V^{\mathbb{P}}$ we mean the Boolean-valued model, but we follow the standard abuse and think of it as $V[G]$ where G is a filter $\mathbb{P}$-generic over V.

A forcing $\mathbb{P}$ is κ-dense iff forcing with $\mathbb{P}$ adds no new λ-sequences of ordinals for any $\lambda < \kappa$.

For κ a cardinal and $\lambda \leq \kappa$ a regular cardinal, $S^\kappa_\lambda = \{\, \alpha < \kappa : \mathrm{cf}(\alpha) = \lambda \,\}$.

NS_κ denotes the ideal of nonstationary subsets of κ.

For finestructure theory, we assume at least some acquaintance with [St1] and references therein. By $0^{\P}$ we mean Schindler's zero-hand-grenade, as defined in [S].

When discussing elementary embeddings $j : M \to N$, M and N are assumed to be transitive unless the membership relation on N is explicitly mentioned. By $\mathrm{cp}(j)$ we mean the critical point of j. A model M is always assumed to be provided with its membership relation (and, possibly, some additional structure), and we in general do not distinguish between a structure and its underlying universe, $M = (M, \in, \dots)$.

For η a cardinal, $H_\eta = \{\, X : \text{ If } Y \text{ is the transitive closure of } X \text{ then } |Y| < \eta \,\}$. We say that the elements of H_η have *hereditary* size less than η. In particular, $H_{\aleph_1}$ is the set of hereditarily countable sets; we also write HC for $H_{\aleph_1}$. We assume that, as a structure, H_η comes equipped with a well-ordering. By $\mathrm{TrCl}\,(x)$ we denote the transitive closure of x.

ZFC^- is $\mathsf{ZFC} - \{\mathsf{Power\ Set}\}$, so $H_\eta \models \mathsf{ZFC}^-$ for all regular $\eta > \omega$.

Let M, N be proper class models of set theory with $M \subseteq N$. Then we call

N an outer model of M, and M an inner model of N. By M_κ, κ an ordinal, we mean $\{x \in M : \mathrm{rk}(x) < \kappa\}$, where $\mathrm{rk}(x)$ is the set-theoretic rank of x. There is an obvious exception to this usage, in that on occasion we refer to a sequence of models by $V_0, V_1, V_2, \ldots$ The notation M_κ is admittedly unfortunate, V_κ^M being perhaps more precise.

If τ denotes a definable object of the universe (either a set or a proper class) and M is a (transitive) model of enough set theory, by τ^M we mean the interpretation of τ inside M. For example, if $x \in M$ then $(\mathcal{P}(x))^M = \mathcal{P}^M(x)$ is the unique $y \subset M$ such that

$$M \models \forall z\, (z \subseteq x \leftrightarrow z \in y).$$

For T an r.e. theory (in the language of set theory), $\mathrm{Con}(T)$ is the usual Π^0_1 rendering of the statement claiming the consistency of T. To claim that two theories T_1 and T_2 are equiconsistent over a third theory T means that Peano Arithmetic PA proves $\mathrm{Con}(T + T_1) \iff \mathrm{Con}(T + T_2)$.

Chapter 2

Projective Well-orderings

The goal of this Chapter is to prove a couple of results illustrating the strength (rather, lack thereof) of the hypothesis that the universe admits a projective well-ordering. The reader should interpret these results as saying that the lack of significant large cardinal structure in the universe allows for pathological well-orderings.

2.1 Strong Cardinals

Our first result is a corollary of computations due to Schindler. References are provided in the proof to follow. Theorem 2.1 was known (independently) to Schindler ([S1]).

Theorem 2.1. *Suppose there is no inner model with ω strong cardinals. Then there is a set forcing extension of V with a projective well-ordering of the reals.*

Proof: The idea of the proof is to obtain a model where CH holds and there is a projectively definable ω_1-sequence of almost disjoint reals, from which by judicious use of almost disjoint forcing we can define a well-ordering of $\mathbb{R}$.

More carefully, we look for a model M of enough set theory containing all the reals and such that

$$M \models \mathsf{CH} + \text{There is a projective well-ordering of } \mathbb{R}.$$

We arrange by our use of almost disjoint forcing that the model M itself is projectively definable, thus obtaining the desired well-ordering.

The model we will work with is $K[r] = L(K \cup \{r\})$, where K is the core model and r is a real. We will arrange things so $K[r] \supseteq \mathrm{HC}$ (and CH holds). That $K[r]$ or, rather, a sufficiently long initial segment of $K[r]$ is projective in the codes if there are no inner models of ω strongs follows from results of Schindler ([S]), Hauser and Hjorth ([HaHj]), and Hauser and Schindler ([HaS]).

Now we proceed to the details.

Suppose that there is no inner model with ω many strong cardinals. Then $0^{\P}$ does not exist[1], and this implies that K does. K is definable, and generically invariant in the sense that for any poset $\mathbb{P}$, $K^V = K^{V^{\mathbb{P}}}$. Let $\mathcal{E}$ be the coherent sequence of extenders of K, $\mathcal{E} = E^K$, so $K = L[\mathcal{E}]$.

Claim 2.2. *Suppose $W \models \beta$ `is strong`, for W a universal weasel (so $\mathrm{ORD}^W = \mathrm{ORD}$). Then*

$$W \models \beta \texttt{ is strong, as witnessed by extenders on } \mathcal{E}.$$

Proof: This is most likely folklore and archeology reveals variations of it in print. See, for example, [St1] Theorem 8.14. For the case that concerns us, the result follows from the argument given in [HaHj], Lemma 1.5, together with the realization that below $0^{\P}$, the references to the measurable Ω can be dispensed with in that proof.

Here is a brief sketch:

Since there are no inner models with a Woodin, $W \models$ `I am iterable`, and there is an embedding $j : K \to W$. Since $K \models V = K$, we may assume $W = K$, and work inside W.

Suppose β is strong, and let $\alpha > \beta^+$ be a cardinal. Let K_1 witness the very soundness of a sufficiently long initial segment of K, say $K\|\alpha$. Let E be an extender witnessing β is strong past α, and consider the ultrapower embedding $\pi_E : K \to \mathrm{Ult}(K, E)$. In virtue of the inductive definition of K, we have that $K^{\mathrm{Ult}(K,E)}\|\alpha = K\|\alpha$. Let $K_2 = \pi_E(K_1)$. Then $K_1\|\alpha = K_2\|\alpha$, and K_2 is a soundness witness for $K\|\beta$ but not for $K\|\beta + 1$.

Compare K_1 and K_2, so they iterate to a common model K^*. Let $\pi_1 : K_1 \to K^*$ and $\pi_2 : K_2 \to K^*$ be the iterations arising from the comparison. Then a standard argument using the definability property shows that $\mathrm{cp}(\pi_1) = \beta$. It follows that in the K_1-to-K^* side

[1] $0^{\P}$, zero-hand-grenade, was introduced in Schindler's habilitationsschrift, published as [S], where finestructure theory is developed (over ZFC) under the assumption that $0^{\P}$ does not exists. Its existence is equivalent to the existence of indiscernibles for a proper class model with a proper class of strong cardinals. For the benefit of the reader familiar with the theory of K "below a Woodin cardinal" à la [St1], but unfamiliar with [S], we explain in Section 2.2 how to modify the argument so only [St1] and the corresponding covering lemma ([MScSt], [MSc]) are required.

of the comparison, an extender with critical point β was used, and the agreement of K_1 and K_2 implies that its length is greater than α.

It follows from the initial segment condition that

$$K_1 \models \beta \texttt{ is strong up to } \alpha\texttt{, as witnessed by extenders on the } E^{K_1} \texttt{ sequence,}$$

and therefore the same holds in K. In effect, let F be the extender with $\text{cp}(F) = \beta$ and $\nu(F) \geq \alpha$ used on the K_1-to-K^* side of the comparison. Then by the agreement of models arising on iteration trees, $F \in E^{K_1}$. The collection of $\eta < \alpha$ such that $(\beta^+)^{K_1} \leq \eta < \alpha$, $\eta = \nu(F\restriction_\eta)$, $F\restriction_\eta$ is not of type Z,[2] and η is a successor ordinal, is cofinal in α. Since $(E^{K_1})_{\eta^*} \neq \emptyset$ implies η^* is a limit ordinal, it follows that for any such η, $F\restriction_\eta$ (rather, its trivial completion) is on the K_1-sequence, with index less than α. $\triangle$

Form now on, by strong we understand strong, as witnessed by extenders on the sequence.

Suppose that there are exactly n strong cardinals in K. We claim that there is a set forcing extension of the universe admitting a $\underset{\sim}{\Delta}^1_{n+3}$-well-ordering of the reals. If V is not closed under sharps (so, in particular, $n = 0$), then in fact the well-ordering can be chosen to be $\underset{\sim}{\Delta}^1_2$, see Section 2.2.

From now on assume V is closed under sharps. Let δ be the largest K-cardinal κ such that $K \models \kappa \texttt{ is strong}$.

Claim 2.3. *There is a strong limit singular cardinal λ such that $\lambda^+ = (\lambda^+)^K$, $\delta < \lambda$, and for all $\kappa < \lambda$, κ is strong in $K\|\lambda$ iff κ is strong in K.*

Proof: By the covering lemma (Theorem 8.18 in [S], which really follows from [MScSt], [MSc]), for any $\beta \geq \omega_2$, $\text{cf}(\beta^{+K}) \geq |\beta|$. In particular, for any singular λ, $\lambda^+ = (\lambda^+)^K$.

Now the result is easy. Let $o(\alpha)$, for α and ordinal, denote the Mitchell order of α in K, so $o(\alpha) \in \text{ORD}$ iff α is not strong. Let $\lambda > \delta$ be strong limit singular and closed under $o\restriction_{\{\beta<\lambda\,:\,K\models\beta \texttt{ is not strong}\}}$. We are done, once we verify that for β strong in K, the lengths of the extenders in $K\|\lambda$ with critical point β are unbounded in λ. But this is clear: If β is strong in K, then it is strong as witnessed by extenders on $\mathcal{E}$, and the argument in Claim 2.2 shows that for any cardinal $\alpha > (\beta^+)^K$, both the lengths and the indices of

[2] See [ScStZ] for the definition of type Z extenders, and the correct statement of the initial segment condition.

extenders on $\mathcal{E}$ witnessing that β is $<\alpha$-strong are cofinal in α. In particular, this holds for λ, so $K\|\lambda \models \beta$ `is strong`. $\triangle$

Fix λ as in the claim. The key to most results involving simply definable well-orderings of the reals is to use the almost disjoint forcing technique of Solovay. This is what we do here, in a finestructural context. First, we need a projectively definable uncountable sequence of almost disjoint reals.

Claim 2.4. *There is a set forcing extension of the universe that collapses λ to ω while preserving $\nu := \lambda^+$ and where, moreover,* $\mathrm{HC} = L_\nu[\mathcal{E}, A]$ *for some $A \subseteq \lambda^+$.*

Proof: By forcing with $\mathtt{Coll}(\nu, 2^\lambda)$ if necessary, we may assume $2^\lambda = \lambda^+$. It follows that we can find $A_1 \subseteq \nu$ such that $L_\nu[A_1] = H_\nu$. Let G be $\mathtt{Coll}(\omega, \lambda)$-generic over V, and work in $V[G]$. Let $A_2 \subseteq \omega$ code G. Notice ν is still a cardinal.

What should be by now a standard argument shows that $\mathrm{HC} = L_\nu[A_1, A_2]$, as an easy consequence of the λ^+-cc of the forcing: Any name for an element of $\mathrm{HC}^{V^{\mathtt{Coll}(\nu,2^\lambda)*\mathtt{Coll}(\omega,\lambda)}}$ appears in an initial segment of $(H_\nu)^{V^{\mathtt{Coll}(\nu,2^\lambda)}} = H_\nu$. $\triangle$

Call V_1 the universe obtained in the claim. We apply the almost disjoint forcing technique to an extension of V_1.

Claim 2.5. *There is a set forcing extension of V_1 preserving $\nu (= \omega_1^{V_1})$ where there is a real r such that* $\mathrm{HC} = L_\nu[\mathcal{E}, r]$.

Proof: Work inside V_1. In K there is a definable sequence of λ^+ subsets of λ. Any reasonable such sequence in fact is definable over $K\|\nu$ and, moreover, for a stationary set of $\gamma < \nu$ the same definition over $K\|\gamma$ gives the first γ-many terms of the sequence.

Let $\hat{s}$ be a real coding λ.[3] Then in $K[\hat{s}]$ we can easily define from the K-sequence an ω_1-sequence of almost disjoint reals (in the usual sense: for any $s_1 \neq s_2$ in the sequence, $|s_1 \cap s_2| < \omega$.)

Let $\mathcal{A} = \langle s_\alpha : \alpha < \omega_1 \rangle$ be such an $L_\nu[\mathcal{E}, \hat{s}]$-definable sequence of almost disjoint reals. Recall that $\mathrm{HC} = L_{\omega_1}[\mathcal{E}, A]$.

Let $\mathbb{Q}$ be the usual forcing for coding A by a real, using $\mathcal{A}$, namely

$$\mathbb{Q} = \{ (s, F) : s \in 2^{<\omega} \text{ and } F \in \mathcal{P}_\omega(\omega_1) \},$$

[3] Usually we only need our objects to be coded in a weak sense, namely, A codes B iff $B \in L(A)$. Here, however, we mean that the natural identification of ω with $\omega \times \omega$ maps s to a well-ordering of ω in order-type λ. By a real we mean here an element of $\mathcal{P}(\omega)$.

ordered by

$$(s, F) \leq (\mathsf{s}, \mathsf{F}) \iff \begin{cases} s \supseteq \mathsf{s}, \\ F \supseteq \mathsf{F} \text{ and} \\ \forall \alpha \in \mathsf{F}\,(\alpha \in A \to (\mathrm{dom}\,(s) \setminus \mathrm{dom}\,(\mathsf{s})) \cap F \cap s_\alpha = \emptyset). \end{cases}$$

It is easy to see, and well known (see for example [Ku] §II.2), that $\mathbb{Q}$ is σ-centered and that if G is $\mathbb{Q}$-generic over V_1, then $r_G = \bigcup\{\, s : \exists F\,((s, F) \in G)\,\} \in (2^\omega)^{V_1[G]}$, G is definable from r_G in $V_1[G]$, and for all $\alpha \in \omega_1$, $\alpha \in A$ iff $|r_G \cap s_\alpha| < \omega$.

Fix such a G, $\mathbb{Q}$-generic over V_1. We claim that $V_1[G]$ is as wanted, and that $r = \{\, 2n : n \in \hat{s}\,\} \cup \{\, 2n+1 : n \in r_G\,\}$ serves as a witness. Namely, $L_{\omega_1}[\mathcal{E}, A] \subset K[r]$, since $\hat{s} \in K[r]$, $\mathcal{A}$ is definable in $K[\hat{s}]$, and A is definable from $\mathcal{A}$ and r_G. Therefore $\mathrm{HC} \subseteq K[r]$, since $\mathbb{Q}$ is ccc. Clearly, $L_{\omega_1}[\mathcal{E}, r] \subseteq \mathrm{HC}$, and we are done. △

Recall that $\delta < \lambda$ be the largest strong cardinal of K. Let V_2 be the universe obtained in the claim, and work in V_2. The following key lemma, pointed out by Schindler ([S2]), improves the complexity of the well-ordering obtained by our original argument.

Lemma 2.6. *$L_\nu[\mathcal{E}]$ is $\Delta^1_{n+3}(s)$ in the codes, where $s \in \mathbb{R}$ codes $K\|\delta$.*

Proof: This follows from Schindler's arguments in [HaS]. See the comment on page 141 of [HaS] after the proof of Theorem 3.5. △

Remark 2.7. Schindler's result is quite impressive. Here it is, in a very general form (There are similar statements for models with infinitely many strong cardinals, but the computations are of course not projective in those cases.)

Theorem 2.8 (Schindler). *Suppose that there is no inner model with a Woodin cardinal and that K exists. Suppose that*

$$\mathfrak{J}^K_{\omega_1^V} \models \text{“}\texttt{There are finitely many strong cardinals}\text{”}.$$

Then $\{\, r \in \mathbb{R} : r$ codes $\mathcal{M} \lhd \mathfrak{J}^K_{\omega_1^V}\,\}$ is projective (we say $\mathfrak{J}^K_{\omega_1^V}$ is projective in the codes[4]*.)*

In fact, suppose

$$\mathfrak{J}^K_{\omega_1^V} \models \text{“}\texttt{There are exactly } n \texttt{ strong cardinals}\text{”}.$$

Then

[4] $\bar{P} \trianglelefteq \bar{Q}$ iff $\bar{P}$ is an initial segment of $\bar{Q}$.

1. *If ω_1^V is inaccessible in K, then $\Game^K_{\omega_1^V}$ is Δ^1_{n+5} in the codes.*

2. *If ω_1^V is a successor in K, then $\Game^K_{\omega_1^V}$ is $\utilde{\Delta}^1_{n+3}$ in the codes.* $\triangle$

Recall that K is generically invariant and that, by results of [HaHj], if there is an inner model with infinitely many strong cardinals in a forcing extension, then K is already a witness. What we did was to ensure the Theorem is applicable, starting from a more relaxed hypothesis.

It now follows that in V_2 the reals admit a $\Delta^1_{n+3}(r,s)$-well-ordering, where r and s are as in the claims. Namely, $K[r] = L(K \cup \{r\})$ so $\mathbb{R}^{V_2} = \mathbb{R}^{K[r]}$ admits a natural well-ordering, derived from the order of constructibility (closing under terms for Gödel operations) and the natural well-ordering of K. More carefully, $K[r] = K[\hat{s}][A][r]$, but $\hat{s}$ is recursive in r and A is easily definable from r and $\mathcal{A}$, which in turn is easily definable in $K[\hat{s}]$ from $\hat{s}$ and a sequence of sets locally definable in $K\|\omega_1$. Unfolding this construction, the terms produced by Gödel operations only require to be (hereditarily) evaluated in elements of $K\|\omega_1$ and the real r. So we obtain a well-ordering by only listing those terms that produce reals, and avoiding repetitions. Since the terms are naturally well-ordered, we only need to see how difficult it is to identify $K\|\omega_1$ inside $K[r]$. Schindler's result tells us that it is $\Delta^1_{n+3}(s)$, and we are done. $\square$

2.2 Projective Well-orderings, Revisited

We indicate how to carry out the argument of Theorem 2.1 for those readers familiar with Steel [St1] but not with Schindler [S]. We must consider two cases, depending on whether the universe is closed under sharps.

Case 1: There is a set X such that $X^\sharp$ does not exist.

By Jensen's covering lemma, there is $\delta > \sup X$ such that $\delta^+ = (\delta^+)^{L[X]}$. Force to make δ countable while preserving δ^+ and let r be a real coding X (r can code a well-ordering of ω in order type δ, and the characteristic function of X as a subset of δ), so $\omega_1 = \omega_1^{L[r]}$.

We are now done. A trivial relativization of Harrington's Theorem 1.13 gives a forcing extension where the reals admit a $\utilde{\Delta}^1_2$-well-ordering.

Case 2: For all X, $X^\sharp$ exists.

Then we can proceed to construct a global K model. The argument for how to carry out this construction is due to Woodin and forms part of his technique known as "the core model induction". It first appeared in print in [Ha].

The idea is to exploit the local definability of K. In [St1], the background certified model K^c is not an inner model, its height being an ordinal Ω. In [St1] it is required of Ω to be measurable in V, but the existence of sharps and even weaker assumptions suffice. What matters is to have at our disposal some form of "external measure", to guarantee the cheapo-covering argument. We give a rough sketch of the argument here, leaving aside some (relevant) issues of iterability.

Granting the existence of sharps for all sets, let κ be a regular cardinal such that $2^{<\kappa} = \kappa$ and let $B \subseteq \kappa$ code H_κ. We can always assume, by passing to a forcing extension if necessary, that there is such a κ. If V is closed under sharps, then so is any set forcing extension. Hence, there is no loss of generality, and we can assume B exists.

Construe $B^\sharp$ as the smallest B-mouse with an active extender. Let Ω_B denote the critical point of the active extender in $B^\sharp$. Build K^c up to Ω_B inside $L[B]$. If the construction halts, then we are done since this implies a non-tame mouse has been reached in $L[B]$ and therefore there are inner models with ω many strong cardinals. Otherwise, the construction succeeds. Then, starting with K^c, we can proceed to build K, which must exist since otherwise there are inner models with ω strongs. Denote K^B the outcome of this construction. Notice $\mathrm{ORD}^{K^B} = \Omega_B$.

Let $B_1 \subseteq \kappa$ be any other set coding H_κ. Inside $L[B_1]$, K^{B_1} can be built up to Ω_{B_1}. Then point of these partial constructions is that they cohere: By the local definability of K (see [St1]), K^B and K^{B_1} coincide past κ, and by considering all the possible sets B, this construction converges to a K-model of height κ^+.

Now continue, by running the same construction but starting with a larger regular cardinal κ_1 such that $2^{<\kappa_1} = \kappa_1$.

The outcome of this inductive procedure is a global K-model. Once we have K at our disposal, the argument given before can proceed as indicated.

2.3 The Strength of Projective Well-orderings

Theorem 2.1 is essentially best possible, as the following corollary indicates. On the other hand, it produces boldface well-orderings, so there is an obvious question whose answer is still missing, see for example Question 2.12 below.

Corollary 2.9. *The following theories are equiconsistent (over* ZFC*):*

1. *There are ω strong cardinals.*

2. *There is no set forcing extension of V with a projective well-ordering of the reals.*

3. *Projective absoluteness.*

Proof: (Con (ZFC + *1.*) $\Rightarrow$ Con (ZFC + *3.*)) By unpublished results of Woodin (See [St], Section 3, specially Corollary 3.7.) if $\lambda = \sup_n \kappa_n$ where $\kappa_0 < \kappa_1 < \dots$ are strong, then projective absoluteness holds in $V_1 = V^{\mathtt{Coll}(\omega,\lambda)}$.

(*3.* $\Rightarrow$ *2.*) Since homogeneous forcing destroys any projective well-ordering, projective absoluteness ensures that no set forcing extension of V_1 can have a projective well-ordering of $\mathbb{R}$.

(Con (ZFC + *2.*) $\Rightarrow$ Con (ZFC + *1.*)) This is immediate from Theorem 2.1. □

The equivalence between *3.* and *1.* in Corollary 2.9 was already known, and it is the content of Hauser's habilitationsschrift, see [Ha]. Our proof of Theorem 2.1, computing the complexity of the well-ordering in terms of the number of strongs in K, includes an improvement due to Schindler ([S2]) of our original computation. We thank him for allowing us to give his sharper, and optimal, result. As mentioned before, Theorem 2.1 itself was known (independently) to Schindler ([S1]). A much deeper property of theory *2.* in the statement of 2.9 above seems untractable with current finestructural techniques.

Question 2.10. *Is it true that projective absoluteness holds iff there is no forcing extension of V with a projective well-ordering of the reals?*

The coding techniques used to prove the theorem allow for natural improvements of the result. For example:

Fact 2.11. *Assume K exists and is not closed under sharps. Then there is a (set) forcing extension of V where the reals admit a $\underset{\sim}{\Delta}^1_2$-well-ordering.* □

This follows from 1.13 and the (easy) observation that if V is closed under sharps, then so is K.

Question 2.12. *Assume $\neg 0^\sharp$. Is there a (set) forcing extension of V with a (lightface) Σ^1_3-well-ordering of $\mathbb{R}$?*

On the other hand, the complexity of the well-orderings obtained in Theorem 2.1 is in general best possible, by Woodin's results on projective absoluteness, see [St].

2.4 Bounded Forcing Axioms

Harrington [Har] showed that MA is consistent with the existence of a $\mathbf{\Sigma}^1_3$-well-ordering of the reals. More precisely, there is a forcing extension of L where MA holds and whose reals admit a well-ordering of the claimed complexity. Harrington's result is optimal in the sense that $\mathbf{\Sigma}^1_2$-well-orderings are incompatible with MA.

This is an immediate corollary of the following two classical results:

Theorem 2.13 (Solovay). *All $\mathbf{\Sigma}^1_2$-sets of reals are Lebesgue measurable iff for all $a \in \mathbb{R}$, the set of reals random-generic over $L[a]$ has full measure.* $\square$

Theorem 2.14 (Martin, Solovay). $\mathsf{MA}_{\aleph_1}$ *implies that for all $a \in \mathbb{R}$, the set of reals random-generic over $L[a]$ has full measure.* $\square$

On the other hand, the Lebesgue measurability of $\mathbf{\Sigma}^1_3$-sets of reals in the context of MA cannot be forced over L unless large cardinals are granted:

Theorem 2.15 (Shelah). *Assume all $\mathbf{\Sigma}^1_3$-sets are Lebesgue measurable. Then $\aleph_1$ is inaccessible from reals: $\aleph_1^{L[a]} < \aleph_1$ for all $a \in \mathbb{R}$.* $\square$

There is a sense in which Harrington's theorem is *not* optimal, namely the well-ordering in [Har] is not lightface. This is not really an obstacle:

Theorem 2.16 (Friedman, see [Fr]). *There is a forcing extension of L that preserves ω_1 in which*

1. MA *holds,*
2. *There is a Σ^1_3-well-ordering of the reals.* $\square$

In this Section we strengthen Harrington's result in a different direction.

Recall:

Definition 2.17 (SPFA($\mathfrak{c}$)). The semiproper forcing axiom holds restricted to posets of size at most $\mathfrak{c}$, i.e., if $|\mathbb{P}| \leq \mathfrak{c}$, $\mathbb{P}$ is semiproper, and $\mathcal{D} \subseteq \mathcal{P}(\mathbb{P})$ is a collection of at most $\aleph_1$ many dense subsets of $\mathbb{P}$, then there is a $\mathcal{D}$-*generic* filter $\mathcal{G} \subseteq \mathbb{P}$, i.e.,

$$\forall D \in \mathcal{D}\,(D \cap \mathcal{G} \neq \emptyset).$$

Definition 2.18. 1. The bounded semiproper forcing axiom, BSPFA, holds iff whenever $\mathbb{P}$ is semiproper and $\mathcal{D}$ is a collection of at most $\aleph_1$ many predense subsets of $\mathbb{P}$, each of cardinality at most $\aleph_1$, then there is a $\mathcal{D}$-generic filter $\mathcal{G} \subseteq \mathbb{P}$.

2. BSPFA^{++} holds iff, with $\mathbb{P}, \mathcal{D}$ as above, if in addition a sequence $\langle \tau_\alpha : \alpha < \omega_1 \rangle$ of $\mathbb{P}$-names for stationary subsets of ω_1 is given, then there is a $\mathcal{D}$-generic filter $\mathcal{G} \subseteq \mathbb{P}$ such that for all $\alpha < \omega_1$,

$$(\tau_\alpha)_\mathcal{G} := \{\, \beta < \omega_1 : \exists p \in \mathcal{G}\,(p \Vdash \beta \in \tau_\alpha) \,\}$$

is stationary in ω_1.

Bagaria [Bag] has found useful equivalent formulations of these principles which show plainly what they can accomplish:

Theorem 2.19 (Bagaria [Bag]). *1.* BSPFA *holds iff $H_{\omega_2} \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{P}}}$ for all semiproper forcings $\mathbb{P}$.*

2. BSPFA^{++} *holds iff $(H_{\omega_2}, \in, \mathrm{NS}_{\omega_1}) \prec_{\Sigma_1} (H_{\omega_2}, \in, \mathrm{NS}_{\omega_1})^{V^{\mathbb{P}}}$ for all semiproper forcings $\mathbb{P}$.* □

Definition 2.20 (Woodin). ψ_{AC} is the following statement: Suppose S and T are stationary, co-stationary subsets of ω_1. Let NS_{ω_1} be the nonstationary ideal on ω_1. Let $\mathbb{P} = \mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. Then there is an $\alpha < \omega_2$ such that whenever G is a $\mathbb{P}$-generic filter over V, then

$$S \in G \quad \text{iff} \quad \alpha \in j(T),$$

where $j : V \to (\mathrm{Ult}(V, G), \tilde{\in}) \subseteq V[G]$ is the generic ultrapower embedding (and, as customary, we identify the standard part of a model with its transitive collapse.)

Definition 2.20 can be restated without mentioning the generic: Given S and T as above, the condition on α is equivalent to

$$[S]_{\mathrm{NS}_{\omega_1}} = [\![\alpha \in j(T)]\!]_{\mathrm{RO}(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})},$$

where j is with Boolean value 1 a $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$-name for the generic ultrapower embedding.

In turn, this is equivalent to stating the existence of a bijection $\pi : \omega_1 \to \alpha$ and of a club $C \subseteq \omega_1$ such that

$$S \cap C = \{\, \beta \in C : \mathrm{ot}(\pi``\beta) \in T \,\}.$$

Theorem 2.21. *Let $L[\mathcal{E}]$ be a finestructural inner model with a strong cardinal and a measurable above but without inner models with Woodin cardinals. Then there is a forcing extension of $L[\mathcal{E}]$ where the following hold:*

1. $\mathsf{SPFA}(\mathfrak{c}) + \mathsf{BSPFA}^{++}$.
2. ψ_{AC}.
3. *There is a Σ^1_5-well-ordering of $\mathbb{R}$.*

Remark 2.22. That a strong cardinal suffices to obtain $\mathsf{SPFA}(\mathfrak{c})$ by forcing was known before, see [W] Remark 2.48. Originally, this was somewhat surprising, given the equivalence between SPFA and MM. Woodin was first to show that $\mathsf{SPFA}(\mathfrak{c})$ is strictly weaker than $\mathsf{MM}(\mathfrak{c})$. For example, [W], Theorem 9.73 states that $\mathsf{MM}(\mathfrak{c})$ implies Projective Determinacy and is therefore in consistency strength strictly above $\mathsf{SPFA}(\mathfrak{c})$. In fact, a strong cardinal is much more than necessary, and we just use it here to speed up the argument.

Proof: The proof divides in a natural way into three parts: First we define for κ a strong cardinal, a revised countable support iteration $\mathbb{P}_\kappa$ of length κ of semiproper forcings, and show that

$$V^{\mathbb{P}_\kappa} \models \mathsf{SPFA}(\mathfrak{c}) + \mathsf{BSPFA}^{++}.$$

Second, if $\xi > \kappa$ is measurable, we show that

$$V^{\mathbb{P}_\kappa} \models \psi_{AC}.$$

This elaborates on an argument of Woodin, Lemma 10.95 of [W]. It does not follow automatically from this lemma, since the forcing axiom we are assuming here is strictly weaker than BMM. From ψ_{AC} it follows that a well-ordering of $\mathbb{R}$ can be easily defined from an infinite sequence of pairwise disjoint stationary sets.

Finally, if $L[\mathcal{E}]$ is as in the hypothesis of the theorem, λ and $\nu > \lambda$ are in $L[\mathcal{E}]$ respectively strong and measurable, and $\mathbb{Q} = (\mathbb{P}_\lambda)^{L[\mathcal{E}]}$, then

$$L[\mathcal{E}] \models \texttt{There is a } \Delta^1_3\texttt{-in the codes sequence } (S_n : n < \omega) \texttt{ of disjoint stationary subsets of } \omega_1.$$

By Σ^1_3-absoluteness between $L[\mathcal{E}]$ and $L[\mathcal{E}]^{\mathbb{Q}}$, such a sequence is still Δ^1_3-definable, and by definition of $\mathbb{Q}$ the S_n are still stationary in ω_1. A calculation then shows that

$$L[\mathcal{E}]^{\mathbb{Q}} \models \texttt{There is a } \Sigma^1_5\texttt{-well-ordering of the reals,}$$

completing the proof.

Remark 2.23. That BSPFA is strictly weaker than BMM is the content of Theorem 3.5 of [AsWe]. Corollary 2.3 of the same paper shows that the statement of [W], Lemma 10.95 can be improved by replacing the measurable with a cardinal κ satisfying the Erdős property $\kappa \to (< \omega_1)^{<\omega}_{2^{\omega_1}}$. See [AsWe] for the relevant definitions. Schindler [S3] shows that even in consistency strength BMM is strictly stronger than BSPFA. Namely, BSPFA is equiconsistent with the existence of Σ_1-reflecting cardinals, while BMM implies that every X belongs to an inner model with a strong cardinal.

Now we proceed to the details:

For revised countable support (RCS) iterations, the reader is advised to consult [DFuc].

Let κ be strong. The key to define $\mathbb{P}_\kappa$ is an appropriate version of Laver functions for strong or locally strong cardinals, due to Shelah and Gitik.

Lemma 2.24 (Gitik, Shelah [GSh1]). *Let κ be strong. Then there is $\ell \vdots \kappa \to V_\kappa$ such that for every x and every $\lambda \geq |\mathrm{TrCl}(x)|$, there is a (κ, λ)-extender E which is λ-strong and such that $j_E(\ell)(\kappa) = x$, where $j_E : V \to \mathrm{Ult}(V, E)$ is the ultrapower embedding given by E.* $\triangle$

Actually, the result in [GSh1] is based on a notion different from that of a (κ, λ)-λ-strong extender: that of a (κ, λ)-normal ultrafilter (as defined in [Bal]) but the proof adapts in a straightforward way.

To understand how the proof goes, it is convenient to compare with the argument given in [L]. It is easy to see that the only use of supercompactness in that proof can be replaced with just strongness. Basically, Laver assumes the result fails, and picks counterexamples for all $f \vdots \kappa \to V_\kappa$. He picks λ_f minimal counterexample to f in the sense that there is x with $\lambda \geq |\mathrm{TrCl}(x)|$, but no extender E witnessing $j_E(f)(\kappa) = x$. Then he considers $j : V \to M$, where j comes from a (κ, λ)-λ-strong extender for λ bigger than all the λ_f. He uses λ-supercompactness to argue that λ_f still witnesses that f does not work in M. But all we need is that M contains enough of V, so it sees that there are no (κ, λ_f)-extenders as required.

Steel prefers to avoid the argument by contradiction and instead arguing directly, defining ℓ inductively. Such an argument is presented in [H1]. This has the disadvantage of requiring a global choice function, but such a predicate can be added by proper class forcing without adding any new sets.

The argument works locally, so appropriate Laver functions exist if κ is only θ-strong for some $\theta \geq \kappa$.

Now we can apply the standard proof of the consistency of SPFA, but working with κ strong: An RCS iteration $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ is defined, so the α^{th} iterand is a $\mathbb{P}_\alpha$-name $\mathbb{Q}_\alpha$ for a semiproper forcing such that $V^{\mathbb{P}_\alpha * \mathbb{Q}_\alpha} \models |\mathbb{P}_\alpha| \leq \aleph_1$, and if $\ell(\alpha) = (\mathbb{P}, \mathcal{D})$, where

- ℓ is our Laver function,
- $\mathbb{P}$ is a $\mathbb{P}_\alpha$-name for a semiproper forcing such that $V^{\mathbb{P}_\alpha * \mathbb{P}} \models |\mathbb{P}_\alpha| \leq \aleph_1$, and
- $\mathcal{D}$ is a $\mathbb{P}_\alpha$ name for a sequence of $\gamma < \kappa$ dense subsets of $\mathbb{P}$,

then $\mathbb{Q}_\alpha$ is defined as $\mathbb{P}$.

Forcing with $\mathbb{P}_\kappa$ is semiproper (See [DFuc], Theorem 4.1) so ω_1 is preserved. It is κ-cc, because at inaccessible points direct limits are taken in RCS iterations (see [J1], Theorem II.7.9.)

Lemma 2.25. $V^{\mathbb{P}_\kappa} \models \mathsf{SPFA}(\mathfrak{c})$.

Proof: The standard proof of the consistency of PFA adapts (See [J1] Theorem III.6.7, [FoMaSh], or [DFuc] Theorem 5.1.) There is only one point that requires elaboration, for its reliance on supercompactness:

Let G be $\mathbb{P}_\kappa$-generic over V. We need to show that

$V[G] \models$ `If` $\mathbb{P}$ `is semiproper,` $|\mathbb{P}| \leq \kappa$, `and` $\mathcal{D}$ `is a sequence of less than` κ`-many dense sets, then there is a` $\mathcal{D}$`-generic filter` $\mathcal{G} \subseteq \mathbb{P}$.

This implies $\kappa = \mathfrak{c} = \aleph_2$ and $\mathsf{SPFA}(\mathfrak{c})$ in $V[G]$, and is shown via a reflection argument. The key is to show that if

- $\mathbb{P}$ is semiproper of size at most κ,
- $\mathcal{D}$ is such a sequence of dense sets,
- $j : V \to M$ is chosen so $j(\ell)(\kappa) =$ ($\mathbb{P}_\kappa$-name for $\mathbb{P} * \mathbb{Q}$, $\mathbb{P}_\kappa$-name for $\mathcal{D}$) where $\mathbb{P} * \mathbb{Q}$ is semiproper, and in the extension by $\mathbb{P} * \mathbb{Q}$, $|\mathbb{P}| \leq \aleph_1$, and
- j is λ-strong (for λ sufficiently bigger than κ),

then $\mathbb{P}$ is still semiproper in $M[G]$.

Jech's argument in [J1] seems to use supercompactness in an essential way. Better, use the characterization of semiproperness in terms of games:

> Let $\mathbb{P}$ be a forcing. I and II play, with I moving first. Player I plays a condition in $\mathbb{P}$, and then they alternate, I playing $\mathbb{P}$-names for countable ordinals, and II playing countable ordinals. II wins iff some condition extending the one I played forces each name to be one of the ordinals played.
>
> $\mathbb{P}$ is semiproper iff II has a winning strategy.

II has a winning strategy in $V[G]$, and the usual reflection argument as in [FoMaSh] works: The relevant embedding $j : V \to M$ can be required to witness enough strength of κ, so it lifts to an embedding $j : V[G] \to M[G^\frown H]$ in the extension $V[G][H]$ (where $G^\frown H$ is $j(\mathbb{P}_\kappa)$-generic over V), with enough agreement still persisting between $M[G^\frown H]$ and $V[G]$ (by κ-cc of $\mathbb{P}_\kappa$.) This shows the winning strategy is in $M[G^\frown H]$ and therefore in $M[G]$—Actually, the winning strategy is a definable class but it suffices to think of it as a function with domain, say, nice names for countable ordinals. A nice name for a countable ordinal is defined from a sequence of antichains. Since $M[G]$ and $V[G]$ can be ensured to agree on what the nice names for countable ordinals are, $M[G]$ has II's winning strategy. $\triangle$

Remark 2.26. The argument given shows that strength is more than is actually needed. The function ℓ only needs to predict small objects, and it is easy to see (2^{2^κ})-strength of κ suffices.

Notice that the argument just given is soft enough that allows for additional clauses, thus providing a method for showing the consistency of $\mathsf{SPFA}(\mathfrak{c})$ together with several other principles. For example, these clauses can be used to implement BSPFA^{++} by copying the argument in the proof of [GoSh], Theorem 2.11, replacing "countable support iteration" with "RCS iteration" and "proper" with "semiproper". Since strong cardinals are Σ_1-reflecting, this argument works. That in fact BSPFA^{++} holds in the extension can be ensured in a straightforward way by adding an additional clause to requirements (1)–(6) in the proof of [GoSh], Theorem 2.11. So, together with the iteration and the sequences of names $\underset{\sim}{\mathfrak{M}}_i$ as defined there, we also have a list of names for ω_1-sequences of names for stationary sets, and estipulate that they are met. We can order our list so each name appears stationarily often, and this guarantees the sufficiently generic filters will guess them as desired. This shows the required result:

Lemma 2.27. $V^{\mathbb{P}_\kappa} \models \mathsf{BSPFA}^{++}$. $\triangle$

This concludes the first part of the proof. For the second one, we only need to show the following:

Lemma 2.28. *Suppose that* BSPFA *holds and* ξ *is a measurable cardinal. Then* ψ_{AC} *holds.*

Proof: This is like [W], Lemma 10.95. Let S and T be stationary, costationary subsets of ω_1. We only need to verify the following: Let $\mathbb{P}$ be the forcing for collapsing ξ to ω_1 via a surjection $\pi : \omega_1 \to \xi$ while shooting a club $C \subseteq \omega_1$ such that

$$T \cap C = \{\, \alpha \in C : \pi``\alpha \in S \,\}.$$

Conditions in $\mathbb{P}$ are closed initial segments, and the order is by extension. Then $\mathbb{P}$ is semiproper.

In effect, let $\eta > \xi$ be sufficiently large and let $X \prec V_\eta$ be countable and contain all relevant parameters. We can assume $X \cap \omega_1 \in T$. Since ξ is measurable, X can be expanded to a structure Y such that $Y \cap \omega_1 = X \cap \omega_1$ yet $\text{ot}(Y \cap \kappa) \in S$. This can be easily achieved by standard arguments. For example, by iterating the construction in Lemma 1.20 of [La1].

With Y as above, if p is the union of a Y-generic chain of conditions (i.e., a descending ω-sequence of conditions in Y meeting every dense set in Y), then $p \cup \{(Y \cap \omega_1, \text{ot}(Y \cap \kappa))\}$ *is* a condition in $\mathbb{P}$ which is clearly X-generic, and semiproperness follows. By our forcing axiom, the instance of ψ_{AC} relevant to S and T must hold. Since they were arbitrary, we are done. $\triangle$

Since we will need this explicit definition, let us now recall how ψ_{AC} can be used to provide us with well-orderings of $\mathbb{R}$.

Suppose $(S_n : n < \omega)$ is a sequence of disjoint stationary subsets ofω_1. We associate to each $x \subseteq \omega$ the set $S_x = \bigcup\{ S_{i+1} : i \in x \}$. Notice S_x is stationary-costationary. The ordinal γ_x is defined from S_x as the least γ such that

$$[S_x]_{\mathrm{NS}_{\omega_1}} = [\![\gamma \in j(S_0)]\!]_{\mathrm{RO}(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})}$$

where j is as before. That γ_x exists is precisely what ψ_{AC} asserts.

Notice that if x and y are distinct sets of numbers, then $\gamma_x \neq \gamma_y$, as $S_x \neq_{\mathrm{NS}_{\omega_1}} S_y$.

Now let $L[\mathcal{E}]$ be as in the hypothesis of the Theorem. We proceed to define in $L[\mathcal{E}]$ a Δ^1_3-in the codes ω-sequence of stationary subsets of ω_1. We show that this sequence is still Δ^1_3-definable in $L[\mathcal{E}]^{\mathbb{Q}}$, and argue that a Σ^1_5-well-ordering of $\mathbb{R}$ can be obtained from it.

First we define in $L[\mathcal{E}]$ the sequence $(S_n : n < \omega)$. We verify they are stationary in ω_1. Since $\mathbb{Q}$ is a revised countable support iteration of semiproper forcings, the stationarity of the sets S_n is preserved when forcing with $\mathbb{Q}$.

Club many $\alpha < \omega_1$ are a local ω_1, $\alpha = \omega_1^{\mathcal{J}^{\mathcal{E}}_\delta}$, where $\mathcal{J}^{\mathcal{E}}_\delta \models \mathsf{ZFC}$. This club exists, by reflection, since (say) there are inaccessibles.

Define β_α as the least ordinal such that ($\beta_\alpha > \alpha$ and) $\mathcal{J}^{\mathcal{E}}_{\beta_\alpha+2} \models \alpha$ `is countable`. For $\gamma \in (\alpha, \beta_\alpha)$, let $\delta_{\gamma,\alpha}$ denote the order type of

$$\{ \mathcal{J}^{\mathcal{E}}_\delta : \gamma < \omega\delta \leq \beta_\alpha,\ \mathcal{J}^{\mathcal{E}}_\delta \models \mathsf{ZFC} \}.$$

Finally, let $\delta_\alpha = \lim_{\gamma \nearrow \beta_\alpha} \delta_{\gamma,\alpha}$. Since $\delta_{\gamma,\alpha}$ decreases as γ increases, δ_α exists.

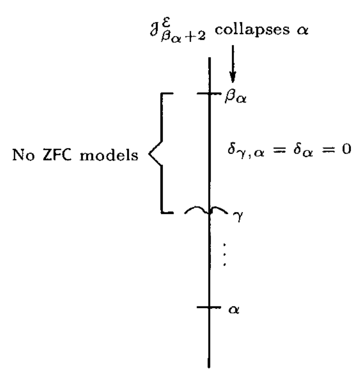

Figure 2.1: Defining S_1

For all α, $\delta_\alpha = 0$ or else it is an additively closed limit ordinal. If $\delta_\alpha = 0$, set $\alpha \in S_1$. If $\delta_\alpha = \omega^n$ where $0 < n < \omega$, set $\alpha \in S_{n+1}$. Otherwise, set $\alpha \in S_0$.

Claim 2.29. *Each S_n is stationary.*

Proof: Suppose S_n is not. Let C be the first club in the order of definability avoiding S_n. Let κ be least such that, setting $\mathcal{M} = \mathcal{J}^{\mathcal{E}}_{\kappa}$, then

- $\mathcal{M} \models \mathsf{ZFC}$,
- $C \in \mathcal{M}$,
- Let $\tau = \text{ot}\{\, \beta < \omega\kappa : \mathcal{M}_\beta \models \mathsf{ZFC} \text{ and } C \in \mathcal{M}_\beta \,\}$. Then either
  - $\tau = 0$ and $n = 1$, or
  - $\tau = \omega^m$, $0 < m < \omega$, and $n = m + 1$, or else
  - $\tau \geq \omega^\omega$ and $n = 0$.

Let $X = \mathtt{Hull}^{\mathcal{M}}(\emptyset)$, so $X \prec \mathcal{M}$, $S_n \in X$ and since κ was chosen so $C \in \mathcal{M}$, then $C \in X$. Let N_X be the transitive collapse of X. Then, by ω-soundness, $N_X = \mathcal{J}^{\mathcal{E}}_{\beta}$ for some countable ordinal β, $N_X \models \mathsf{ZFC}$ and N_X is pointwise definable (without parameters). Let $\alpha = \omega_1 \cap X = \omega_1^{N_X}$. Then α is countable in $\mathcal{J}^{\mathcal{E}}_{\beta+2}$ and therefore $\beta = \beta_\alpha$.

By the requirement on τ and minimality of κ, it follows that $\alpha \in S_n$. But then we obtain a contradiction because $C \in X$, so $\alpha \in C$. $\triangle$

It follows from the mouse condition that $(\, S_n : n < \omega \,)$ is Δ^1_3-in the codes: $\alpha \in S_n$ iff there is a real x coding n and either the finite ordinal α (if $\alpha < \omega$) or a well-ordering of ω in order-type α, and there is a real y coding a mouse $\mathcal{M}$ of the appropriate kind such that $\alpha \in \mathcal{M}$ and $\mathcal{M}$ serves as a witness to the membership of α in S_n. Equivalently, $\alpha \in S_n$ iff for every such x and every y coding such an $\mathcal{M}$, $\mathcal{M}$ certifies this membership.

Remark 2.30. The same argument produces in L a Δ^1_2 such partition. The complexity increases once $L[\mathcal{E}]$ admits Woodin cardinals.

Claim 2.31. *$(\, S_n : n < \omega \,)$ is Δ^1_3-in the codes in $L[\mathcal{E}]^{\mathbb{Q}}$.*

Proof: This is a consequence of Σ^1_3-absoluteness between the ground model and its forcing extension. The sequence is defined by two formulas ψ_0 and ψ_1, where ψ_0 and $\neg\psi_1$ are Σ^1_3.

The formulas express the Σ^1_3-predicates described above, together with the clause that the mouse $\mathcal{M}$ coded by the real y is ω-sound.

Absoluteness shows that the formulas still define the same sequence in the extension. For any $\alpha < \omega_1^{L[\mathcal{E}]}$ and any $n < \omega$, there is a mouse in $L[\mathcal{E}]$ witnessing $\alpha \in S_n$ iff there is such a mouse in $L[\mathcal{E}]^{\mathbb{Q}}$. Since ω_1 is preserved in the extension, we are done. $\triangle$

Now we use this sequence and argue from ψ_{AC} that a Δ^1_5-well-ordering of the reals can be defined. The well-ordering is simply

$$x < y \quad \text{iff} \quad \gamma_x < \gamma_y,$$

where $x \mapsto \gamma_x$ is as defined above.

Notice that $\mathbb{R}^{L[\mathcal{E}]}$ has size ω_1 and is a Σ^1_3 set in $L[\mathcal{E}]^{\mathbb{Q}}$. This is a consequence of absoluteness, considering a good Δ^1_3-well-ordering of $\mathbb{R}$ in $L[\mathcal{E}]$, see the proof of [W], Theorem 3.28. By $\mathsf{SPFA}(\mathfrak{c})$, we can talk about subsets of ω_1 using this sequence and almost disjoint forcing. We can thus define the well-ordering by saying that $x < y$ iff (with respect to this sequence of reals) there are codes for a club set C and for bijections $\pi_1 : \omega_1 \longrightarrow \gamma_x$, $\pi_2 : \omega_1 \longrightarrow \gamma_y$ of ω_1 into *minimal* ordinals satisfying the requirements of the definitions of γ_x, γ_y with respect to the sequence of S_n, namely,

$$S_x \cap C = \{\, \alpha \in C : \operatorname{ot}(\pi_1``\alpha) \in S_0 \,\}$$

and

$$S_y \cap C = \{\, \alpha \in C : \operatorname{ot}(\pi_2``\alpha) \in S_0 \,\}$$

and such that $\gamma_x < \gamma_y$.

This is a Σ^1_5-definition. $\square$

Remark 2.32. A careless previous version of the result made the strengthened claim that something like the above gives a Σ^1_4-well-ordering. However, this seems difficult to achieve by set forcing, given that Σ^1_3-absoluteness holds: It is conceivable that for $(\, S_n : n < \omega \,)$ as above and for some real x, there is $\hat{\gamma}_x < \gamma_x$ such that for some stationary costationary set T,

$$[S_x \cup T]_{\mathrm{NS}_{\omega_1}} = [\![\hat{\gamma}_x \in j(S_0)]\!]_{\mathrm{RO}(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})}.$$

If such is the case, it looks like the value of γ_x could be lowered at least to $\hat{\gamma}_x$ by shooting a club that misses T. But Σ^1_3-absoluteness would seem to prevent this from happening, if the well-ordering were Σ^1_4.

This does not mean that a Σ^1_4-well-ordering in the statement of Theorem 2.21 is impossible, but maybe class forcing techniques are required, the problem becoming that of adding solutions to a projective (Π^1_3) predicate via projective (Π^1_3) singletons.

Question 2.33. *Is the existence of Σ^1_4-well-orderings of $\mathbb{R}$ consistent with* $\mathsf{SPFA}(\mathfrak{c}) + \mathsf{BSPFA}^{++} + \psi_{AC}$*?*

Chapter 3

Real-valued Measurable Cardinals

Suppose $\mathfrak{c}$ is real-valued measurable. In this Chapter we show that no well-ordering of $\mathbb{R}$ belongs to $L(\mathbb{R})$. However, it is consistent, but not provable, that $\mathbb{R}$ admits third-order definable well-orderings. Specifically, we provide a general argument that produces a model where $\mathfrak{c}$ is real-valued measurable, and there is a Σ^2_2-well-ordering of $\mathbb{R}$. A variation of this idea gives Σ^2_1-well-orderings when applied to $L[\mu]$ or, more generally, $\Sigma^2_1(\Gamma^\infty)$,[1] provided enough large cardinals exist in V, if applied to nice inner models. Recent results of Woodin indicate how to transform this result into a proof from large cardinals of the Ω-consistency of real-valued measurability of $\mathfrak{c}$ together with the existence of Σ^2_1-definable well-orderings of $\mathbb{R}$. It follows that if the Ω-conjecture is true, and large cardinals are granted, then this statement can always be forced.

However, a strengthening of real-valued measurability (real-valued hugeness) is introduced, shown consistent, and shown to contradict the existence of any third-order definable well-orderings of $\mathbb{R}$ at all.

3.1 Basics

This Section is included in order to make this Chapter of the thesis reasonably self-contained, and we do not claim much originality here, other than by way of exposition. The main references for the theory of real-valued measurable cardinals are [So] and [F1], see also [Ku1] and [GSh]. For whatever modest contributions in this Section are due to us, see after Fact 3.24. We start by defining our basic objects:

[1] See the discussion and definitions following the end of the proof of Theorem 3.68.

Definition 3.1. A cardinal κ is *real-valued measurable*, $\mathsf{RVM}(\kappa)$, iff there is a κ-additive continuous[2] probability measure ν with domain $\mathcal{P}(\kappa)$. We call ν a *witnessing* probability.

A real-valued measurable cardinal κ is *atomlessly measurable* iff there is an atomless witnessing probability ν.

The following is due to Ulam ([U]), who also introduced the concept:

Theorem 3.2. *If* $\mathsf{RVM}(\kappa)$*, then* κ *is either measurable or atomlessly measurable, in which case* $\kappa \leq \mathfrak{c}$. $\square$

Definition 3.3. Let ν be a complete measure on some set X. Then

$$\mathcal{N}_\nu := \{\, Y \subseteq X : \nu(Y) = 0 \,\}$$

is the ideal of *ν-null sets*.

Since $\mathrm{add}(\mathcal{N}_\nu)$ is necessarily a regular cardinal, we have the following useful fact:

Fact 3.4. *Suppose* $\mathsf{RVM}(\kappa)$ *and* ν *is a witnessing probability. Then:*

1. $\kappa = \mathrm{add}(\mathcal{N}_\nu)$ *is regular.*

2. $\mathcal{N}_\nu$ *is an* $\aleph_1$*-saturated ideal on* κ. $\square$

Remark 3.5. In fact, if $\kappa \leq \mathfrak{c}$ is real-valued measurable, then κ is weakly Mahlo, the κ^{th} weakly Mahlo, etc.

The following basic characterization is due to Solovay, and will be essential for our arguments:

Theorem 3.6. $\mathsf{RVM}(\kappa)$ *iff there is* $\lambda \geq \omega$ *such that*

$$V^{\mathtt{Random}_\lambda} \models \exists j : V \overset{\prec}{\longrightarrow} N, \quad \mathrm{cp}(j) = \kappa,$$

where $\mathtt{Random}_\lambda$ *is the forcing for adding* λ *many random reals.*

Specifically, $\mathtt{Random}_\lambda$ is the collection of Borel subsets of 2^λ, modulo null sets, where the measure φ is defined as follows:

[2]See Definition 1.1.

- For $J \subset \lambda$, J finite, and $z \in 2^J$, the *cylinder* determined by J, z is

$$C := \{ x \in 2^\lambda : x\restriction_J = z \}.$$

For such a C, define $\varphi(C) := 2^{-|J|}$.

- The cylinders generate the product topology on 2^λ. Extend φ to a Borel measure by:

$$\varphi(B) := \inf\{ \sum_n \varphi(C_n) : B \subseteq \bigcup_n C_n,\ C_n \text{ a cylinder} \}$$

for B a Borel subset of 2^λ.

Remark 3.7. In fact, we can extend φ to a complete measure in the standard way. Some presentations of Random forcing assume that we are working with this completion and not just with its restriction to Borel sets. For the purposes of forcing, the resulting Boolean algebras are equivalent, and we can ignore the difference.

Definition 3.8. Let $\mathbb{B}$ be a σ-complete Boolean algebra. A *'probability measure'* on $\mathbb{B}$ is a function $\nu : \mathbb{B} \to [0, 1]$ such that

1. $\nu(a) = 0$ iff $a = \mathsf{0}$.

2. $\nu(\mathsf{1}) = 1$.

3. ν is σ-additive: If $\{ a_n : n \in \omega \}$ is an antichain in $\mathbb{B}$, so $a_n \cdot a_m = \mathsf{0}$ whenever $n \neq m$, then

$$\nu(\sum_n^{\mathbb{B}} a_n) = \sum_n \nu(a_n).$$

We call $(\mathbb{B}, \nu)$ a *measure algebra.*

Fact 3.9. *1. For all λ, $\mathtt{Random}_\lambda$ is ccc. Thus $\mathtt{Random}_\lambda$ is a complete Boolean algebra.*

2. The map $\nu : \mathtt{Random}_\lambda \to [0, 1]$ given by $\nu([X]) = \varphi(X)$, where φ is as described above and $[X]$ denotes the equivalence class of the Borel subset $X \subseteq 2^\lambda$, is a 'probability measure', so $(\mathtt{Random}_\lambda, \nu)$ is a measure algebra. □

Remark 3.10. Given any probability space $(X, \mathcal{P}, \mu)$, $\mathcal{P}/\mathcal{N}_\mu$ can be turned into a measure algebra by exactly the same construction as in *2.* of the fact.

More significantly,

Fact 3.11. *Any measure algebra is isomorphic (as measure algebra) to one of the form* $\mathcal{P}/\mathcal{N}_\mu$ *for some probability space* $(X, \mathcal{P}, \mu)$, *where* $\mathcal{P}/\mathcal{N}_\mu$ *is a measure algebra with the 'probability measure' described in Fact 3.9.2.* □

This is a consequence of the so called Loomis-Sikorski theorem (due to von Neumann) stating that any σ-complete Boolean algebra is isomorphic (as a Boolean algebra) to $\Sigma/\mathcal{J}$ for some σ-algebra Σ of subsets of some set X, and some σ-complete ideal $\mathcal{J}$ of Σ. See [Kop] and [F] for details.

Definition 3.12. 1. For $\mathbb{B}$ a complete Boolean algebra, the *generating number* of $\mathbb{B}$ is $\tau(\mathbb{B}) := \min\{\,|X| : X$ generates $\mathbb{B}$ (as a complete algebra)$\,\}$.

2. $\mathbb{B}$ is τ*-homogeneous* iff[3] $\tau(\mathbb{B}) = \tau(\mathbb{B}\restriction_a)$ for any $a \neq \mathtt{0}$.

Fact 3.13. *1. If* $\mathbb{B}$ *is a complete Boolean algebra which is homogeneous in the forcing sense*[4], *then* $\mathbb{B}$ *is* τ*-homogeneous.*

2. $\mathtt{Random}_\lambda$ *is homogeneous. Thus, it is* τ*-homogeneous, and* $\tau(\mathtt{Random}_\lambda) = \lambda$. □

Theorem 3.14 (Maharam). *If* $\mathbb{B}$ *is a complete* τ*-homogeneous measure algebra, then it is isomorphic as a measure algebra to exactly one* $\mathtt{Random}_\lambda$ *up to the cardinality of* λ. □

Maharam's theorem is actually much more general than we have stated, but this particular case is all we need.

Fact 3.15. *If* $\mathbb{B} \lessdot \mathtt{Random}_\lambda$ *(i.e.,* $\mathbb{B}$ *is a complete subalgebra of* $\mathtt{Random}_\lambda$*), then* $\mathbb{B} \cong \mathtt{Random}_\gamma$ *for some* γ. □

Notice that, conversely, if $\gamma < \lambda$, then $\mathtt{Random}_\gamma \lessdot \mathtt{Random}_\lambda$.

Remark 3.16. The version of Fact 3.15 for Cohen forcing is false for $\lambda \geq \omega_2$. There are different ways of stating this result. For example, the following is the main theorem of [KopSh]:

Definition 3.17. For $p \in \mathbb{B} \setminus \{\mathtt{0}\}$, let $\mathbb{B}\restriction_p$ be the Boolean algebra of elements of $\mathbb{B}$ below p.

A Boolean algebra $\mathbb{B}$ has *uniform density* κ iff for every nonzero condition $p \in \mathbb{B}$ there is a dense subset of size κ in $(\mathbb{B}\restriction_p) \setminus \{\mathtt{0}\}$.

[3] See Definition 3.17.

[4] I.e., weakly homogeneous in the sense of Kunen [Ku].

Definition 3.18. For a cardinal κ, let $\mathbb{C}_\kappa$ be the Boolean completion of $\mathtt{Add}(\omega, \kappa)$, the forcing for adding κ-many Cohen reals. A Boolean algebra $\mathbb{B}$ is a *standard Cohen algebra* iff it is isomorphic to some $\mathbb{C}_\kappa$.

$\mathbb{B}$ is a *Cohen algebra* iff it satisfies the countable chain condition and every $\mathbb{B}$-generic forcing extension of the universe arises from forcing with some standard Cohen algebra.

It is easy to see that $\mathbb{B}$ is a Cohen algebra iff its completion is isomorphic to a product of at most countably many standard Cohen algebras.

Theorem 3.19 (Koppelberg, Shelah [KopSh]). *If $\kappa \geq \aleph_2$, then there is a complete regular subalgebra of $\mathbb{C}_\kappa$ of uniform density κ which is not isomorphic (in the sense of forcing) to a Cohen algebra.* □

On the other hand, all such subalgebras of $\mathbb{C}_\omega$ are isomorphic to $\mathbb{C}_\omega$, and the same holds for $\kappa = \omega_1$ (See [Kop1] or [BJZ].)

Fact 3.20. *If $W \supseteq V$ is an outer model and G (identified as a subset of λ) is $(\mathtt{Random}_\lambda)^W$-generic over W, then G is $(\mathtt{Random}_\lambda)^V$-generic over V. In particular, for any $\mathbb{P}$, $\mathtt{Random}_\lambda$ completely embeds into $\mathbb{P} * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for $(\mathtt{Random}_\lambda)^{V^{\mathbb{P}}}$.* □

This follows for the results of [Ku1], §3.

Proof of Solovay's Theorem ($\Leftarrow$) Suppose

$$V^{\mathtt{Random}_\lambda} \models \exists j : V \xrightarrow{\prec} N, \quad \mathrm{cp}(j) = \kappa.$$

Let $\varphi : \mathtt{Random}_\lambda \to [0,1]$ be the 'probability measure' associated to $\mathtt{Random}_\lambda$. In V, we want to define a probability measure on subsets of κ. Fix a name j such that

$$[\![\exists N\, (j : V \xrightarrow{\prec} N, \mathrm{cp}(j) = \kappa)]\!] = 1.$$

For $A \subseteq \kappa$, let $\nu(A) := \varphi[\![\kappa \in j(A)]\!]$, so $\nu : \mathcal{P}(\kappa) \to [0,1]$. It is easy to verify that ν is as wanted[5].

($\Rightarrow$) Suppose $\mathsf{RVM}(\kappa)$. Let ν be a witness, and let $\mathbb{B}_\nu = \mathcal{P}(\kappa)/\mathcal{N}_\nu$. Since $\mathcal{N}_\nu$ is $\aleph_1$-saturated, $\mathbb{B}_\nu$ is complete, and we may assume (by reducing to a subset if necessary)

[5]Those uncomfortable with our use of proper classes are advised to consult [So], where the appropriate first-order formulation and proof of Solovay's theorem can be unearthed.

that $\mathbb{B}_\nu \cong \mathtt{Random}_\lambda$ for some infinite λ (Necessarily, for some $X \subseteq \kappa$, $X \notin \mathcal{N}_\nu$, we must have that $\mathcal{P}(X)/\mathcal{N}_\nu$ is τ-homogeneous. By replacing ν with $\hat{\nu} : Y \mapsto \nu(X \cap Y)$, we may as well assume $X = \kappa$). Notice that λ as chosen here is always uncountable.

Let G be $\mathbb{B}_\nu$-generic over V. Then G is essentially a V-ultrafilter on κ, and we can form $\pi : V \to \mathrm{Ult}(V, G)$ in $V[G]$. But the saturation of $\mathcal{N}_\nu$ ensures that the ultrapower is well founded, and therefore isomorphic to a transitive class N. Let $j : V \xrightarrow{\prec} N$ denote the corresponding embedding, coming from π via the Mostowski collapse. Then $\mathrm{cp}(j) = \kappa$, and since $\mathbb{B}_\nu \cong \mathtt{Random}_\lambda$, we are done. □

Fact 3.21. *Suppose* $\mathsf{RVM}(\kappa)$ *and* $\mathtt{Random}_\lambda$, j *and* N *are as in Solovay's theorem. Then* $\mathbb{R}^N = \mathbb{R}^{V^{\mathtt{Random}_\lambda}}$.

Proof: This is standard from the theory of saturated ideals: In fact, using the notation from the theorem, if G is $\mathbb{B}_\nu$-generic over V, then $V[G] \models {}^\omega N \subseteq N$.[6] □

Remark 3.22. Suppose $\mathsf{RVM}(\mathfrak{c})$ and ν is a witness such that $\mathcal{P}(\mathfrak{c})/\mathcal{N}_\nu$ is homogeneous. As explained, it follows that $\mathcal{P}(\mathfrak{c})/\mathcal{N}_\nu \cong \mathtt{Random}_\lambda$ for some λ. It is a result of Gitik and Shelah that $\lambda = 2^{\mathfrak{c}}$.

Solovay's characterization allows for easy proofs of several results of the classical theory of real-valued measurability. For example:

Corollary 3.23 (Silver). *If* $\mathsf{RVM}(\kappa)$ *then the tree property holds for* κ.

Proof: Suppose $\mathsf{RVM}(\kappa)$, and let ν be a witnessing probability. Suppose $\mathcal{T}$ is a κ-tree. Without loss, $\mathcal{T} = (\kappa, <_\mathcal{T})$. As usual, we will identify $\mathcal{T}$ or its levels $\mathcal{T}_\alpha$, $\alpha < \kappa$, with the underlying subsets of κ. Our convention is that trees grow upward, so if 0 is the root of $\mathcal{T}$, $0 <_\mathcal{T} a$ for any other $a \in \mathcal{T}$, etc. Let λ be such that in $V^{\mathtt{Random}_\lambda}$ there is $j : V \to N$ with $\mathrm{cp}(j) = \kappa$. Work in $V^{\mathtt{Random}_\lambda}$.

[6] A strong version of this result is that without loss of generality, the null ideal $\mathcal{N}_\nu$ is normal, so the identity represents κ in the ultrapower N (This only simplifies notation in what follows and is not essential. See [F1] Theorem 1G for a proof.) Given any term $\langle \tau_\alpha : \alpha < \kappa \rangle$ for a κ-sequence in $V[G]$ of elements of N, there is in V a sequence $\langle f_\alpha : \alpha < \kappa \rangle$ of functions, $f_\alpha : \kappa \to V$ for all $\alpha < \kappa$, such that $[\![[f_\alpha]_N = \tau_\alpha]\!] = 1$ (This requires some argument.) Letting $g : \kappa \to V$ be the function given by $g(\beta) = \langle f_\alpha(\beta) : \alpha < \beta \rangle$ for all $\beta < \kappa$ then, in $V[G]$, $[g]_N = j(g)(\kappa) = (\lambda\beta . j\langle f_\alpha : \alpha < \kappa \rangle \restriction_\beta(\beta))(\kappa) = j\langle f_\alpha : \alpha < \kappa \rangle \restriction_\kappa(\kappa) = \langle j(f_\alpha)(\kappa) : \alpha < \kappa \rangle = \langle [f_\alpha]_N : \alpha < \kappa \rangle$. Hence, ${}^\kappa N \subseteq N$. In particular, $\mathcal{P}^{V[G]}(\kappa) \subseteq N$.

Then $j(\mathcal{T})\restriction_\kappa = \mathcal{T}$. Let $\mu = j(\nu)$, so μ witnesses $\mathsf{RVM}(j(\kappa))$ inside N. For $\alpha < j(\kappa)$, let $A_\alpha = \{\beta : \alpha <_{j(\mathcal{T})} \beta\}$. Since μ is $j(\kappa)$-complete, $\mu(\mathcal{T}) = 0$ and there is some $\alpha \in j(\mathcal{T})_\kappa$ such that $\mu(A_\alpha) > 0$.

Let $b = \{\beta \in \mathcal{T} : \beta <_{j(\mathcal{T})} \alpha\}$, and let $\langle b_\gamma : \gamma < \kappa\rangle$ be its $<_\mathcal{T}$-increasing enumeration. Then $\mu(A_{b_\gamma}) \geq \mu(A_{b_\rho})$ whenever $\gamma < \rho$. Since $\kappa > \omega$, for some $\rho < \kappa$ we must have $\mu(A_{b_\rho}) = \mu(A_{b_\tau})$ for all $\tau > \rho$.

For $\beta < \kappa$, $b_\rho <_\mathcal{T} \beta$, let $B_\beta = \{\gamma < \kappa : \beta <_\mathcal{T} \gamma\}$. Notice that $\mu(A_\beta) = j(\nu(B_\beta)) = \nu(B_\beta)$ for any such β. Let $\varepsilon = \nu(B_{b_\rho})$. Then $\forall \beta\ {}_\mathcal{T}\!> b_\rho$, either $\nu(B_\beta) = \varepsilon$, or $\nu(B_\beta) = 0$ (If $0 < \nu(B_\beta) < \varepsilon$, and $\beta \in \mathcal{T}_\gamma$, then $\beta \neq b_\gamma$ and $B_\beta \cap B_{b_\gamma} = \emptyset$. But then $\nu(B_{b_\gamma}) \leq \nu(B_{b_\rho} \setminus B_\beta) < \varepsilon$, a contradiction.)

Let $\underset{\sim}{b} = \{\beta : \beta \leq_\mathcal{T} b_\rho \text{ or } (b_\rho <_\mathcal{T} \beta \text{ and } \nu(B_\beta) = \varepsilon)\}$. Then $b = \underset{\sim}{b} \in V$ is a κ-branch through $\mathcal{T}$. □

One of the main results on preservation of real-valued measurability is the following:

Fact 3.24 (Solovay). *Suppose* $\mathsf{RVM}(\kappa)$. *Then κ stays real-valued measurable after forcing with any* $\mathtt{Random}_\lambda$ *or more generally (by Maharam's theorem), with any measure algebra.*
□

We now argue that if $\mathbb{P}$ is ccc, $\lambda \geq \omega$, and $\mathbb{F} = \mathtt{Random}_\lambda$, then $\mathbb{P}$ is still ccc in $V^\mathbb{F}$.

Definition 3.25. $\mathbb{Q} \in V$ is *absolutely ccc* iff for all outer models $W \supseteq V$, $W \models \mathbb{Q}\ \mathtt{is\ ccc}$.[7]

For example $\mathtt{Add}(\omega, 1)$, $\mathtt{Coll}(\omega, < \omega_1)$, and any σ-centered poset are absolutely ccc. The class of absolutely ccc posets is closed under finite support products and finite support iterations[8]. The following example is slightly more interesting, and we will have several occasions to use it.

Claim 3.26. *All measure algebras, in particular all* $\mathtt{Random}_\lambda$, *are absolutely ccc.*

Proof: Let $\mathbb{P} = (\mathbb{B}, \nu) \in V$ be a measure algebra, and let $W \supseteq V$ be an outer model. Let $\omega_1 = \omega_1^W$.

[7]This definition takes place in MK. For a ZFC rendering, restrict the outer models to those of the form $V^\mathbb{F}$ for $\mathbb{F} \in V$ a poset.

[8]For products, this follows from [Ku] Theorem II.1.9. Since ccc is preserved under finite suport iterations, the result for iterations follows easily from the definition of absolutely ccc, because if $\mathbb{P} \in V$ is the finite support iteration of a family $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \right\rangle$, then in any outer model $W \supseteq V$, $\mathbb{P}$ densely embeds into the finite support iteration of the $\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha$, $\alpha < \lambda$.

Suppose in W that $\langle b_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-antichain in $\mathbb{B} \setminus \{0\}$. Then we can assume that for some $n > 0$, $\nu(b_a) > \frac{1}{n}$ for all α. This is a contradiction: For any $N \in \mathbb{N}$ the sequence $\langle b_m : m < N \rangle$ is in V and since the b_α form an antichain, we have that $\nu(\sum^{\mathbb{B}}_{m<N} b_m) = \sum_{m<N} \nu(b_m) > \frac{N}{n} > 1$ if N is sufficiently large. □

Claim 3.27. *If $\mathbb{P}$ is ccc and $\mathbb{Q}$ is absolutely ccc, then* $V^{\mathbb{Q}} \models \mathbb{P}$ `is ccc`.

Proof: Since $\mathbb{P} \times \mathbb{Q} \cong \mathbb{P} * \check{\mathbb{Q}} \cong \mathbb{Q} * \check{\mathbb{P}}$, it suffices to see that $V^{\mathbb{P}} \models \mathbb{Q}$ `is ccc`, but this holds by hypothesis. □

Corollary 3.28. *Let* $\mathbb{F} = \mathtt{Random}_\lambda$ *for $\lambda \geq \omega$, and let $\mathbb{P}$ be ccc. Then $\mathbb{P}$ is ccc in $V^{\mathbb{F}}$.* □

Corollary 3.29. *The existence of atomlessly measurable cardinals is independent of the existence of Suslin trees.*

Proof: Let κ be measurable, and suppose S is a Suslin tree. Then

$$1 \Vdash_{\mathtt{Random}_\kappa} \mathsf{RVM}(\mathfrak{c}) \text{ and } S \text{ is ccc},$$

by Corollary 3.28. Thus, $V^{\mathtt{Random}_\kappa} \models$ `There is a Suslin tree`.

The other direction is immediate from a result of Laver (see [BJ], Theorem 3.2.31.) Namely, if $\mathsf{MA}_{\aleph_1}$ holds then for any κ,

$$V^{\mathtt{Random}_\kappa} \models \texttt{Every Aronszajn tree is special.}$$

In particular, if κ is measurable and MA holds, then $V^{\mathtt{Random}_\kappa}$ is a model of $\mathsf{RVM}(\mathfrak{c})$ where there are no Suslin trees. □

Stronger versions of the following theorem can be obtained, but this suffices for our purposes. Notice the particular case where κ is measurable, so $\mathbb{B}_\nu$ is trivial and $G \in V$.

Theorem 3.30. *Suppose* $\mathsf{RVM}(\kappa)$ *and let ν be such that $\mathbb{B}_\nu = \mathcal{P}(\kappa)/\mathcal{N}_\nu$ is homogeneous. Let G be $\mathbb{B}_\nu$-generic over V, and in $V[G]$ let $j : V \to N$ be the associated generic embedding. Then the forcing* $j(\mathtt{Random}_\kappa)/\mathtt{Random}_\kappa$ *is isomorphic to* $\mathtt{Random}_{j(\kappa)}$.

Proof: In N, $j(\mathtt{Random}_\kappa) = \mathtt{Random}_{j(\kappa)}$, so $\mathtt{Random}_\kappa \lessdot j(\mathtt{Random}_\kappa)$, and the quotient forcing makes sense. Let $\underline{H}$ be the canonical $\mathtt{Random}_\kappa$ name for the generic filter and recall that, by definition, $j(\mathtt{Random}_\kappa)/\mathtt{Random}_\kappa$ is (a $\mathtt{Random}_\kappa$ name for) the forcing

$$\mathbb{P} = \{\, q \in j(\mathtt{Random}_\kappa) : q \text{ is compatible with every } p \in \underline{H} \,\}.$$

Consequently, fix H a $\mathtt{Random}_\kappa$ generic over V and therefore over N, and work in $V[H]$.

- *In* $N[H]$, $\mathbb{P} \cong \mathtt{Random}_{j(\kappa)}$.

  This is well known, and follows from [Kul], Theorem 3.13.

- *In* $V[H]$, $\mathbb{P}$ *is a* σ*-complete homogeneous boolean algebra.*

  Recall that ${}^\omega N \subset N$, and therefore (by the ccc of $\mathtt{Random}_\kappa$) ${}^\omega N[H] \subset N[H]$, from which σ-completeness in $V[H]$ follows. Homogeneity is clear, since $\mathbb{P}$ is already homogeneous in $N[H]$.

- *In* $V[H]$, $\mathbb{P}$ *is a complete measure algebra.*

  The 'probability measure' witnessing $\mathbb{P}$ is a measure algebra in $N[H]$ is a 'probability measure' in $V[H]$, since $N[H]$ is closed under ω-sequences. Hence, $\mathbb{P}$ is a measure algebra. It is ccc, by Claim 3.26. Completeness follows.

- *In* $V[H]$, $\mathbb{P}$ *is isomorphic to some* $\mathtt{Random}_\rho$ *and, in fact,* $\mathbb{P} \cong \mathtt{Random}_{|j(\kappa)|}$.

  This follows now from Maharam's theorem.

This completes the proof. $\square$

For a generalization, see the first paragraph of the proof of Claim 3.35.

Theorem 3.30 will prove useful in the following sections, where we obtain the consistency of a third-order definable well-ordering of $\mathbb{R}$ together with real-valued measurability of the continuum. That we cannot improve this result in a straightforward fashion is the content of the following result.

Theorem 3.31. *If* $\mathsf{RVM}(\kappa)$ *then no well-ordering of* $\mathbb{R}$ *belongs to* $L(\mathbb{R})$.

Proof: This is standard. Assume by contradiction that $\mathsf{RVM}(\kappa)$ and there is $\varphi(x,y,z,w)$ a formula in the language of $L(\mathbb{R})$ such that for some real t and ordinal α, the relation between reals

$$r < s \quad \Longleftrightarrow \quad L(\mathbb{R}) \models \varphi(r,s,t,\alpha)$$

is a well-ordering of $\mathbb{R}$. The least such α is definable in $L(\mathbb{R})$, so there is such a formula φ' all of whose parameters are reals. Let λ be as above, so in $V^{\mathrm{Random}_\lambda}$ there is an embedding $j : V \xrightarrow{\prec} N$ such that $\mathrm{cp}(j) = \kappa$ and ${}^\omega N \subseteq N$. Then

$$j\restriction_{L(\mathbb{R})^V} : L(\mathbb{R})^V \xrightarrow{\prec} L(\mathbb{R})^{V^{\mathrm{Random}_\lambda}}.$$

In particular, there is $t \in \mathbb{R}^V$ such that $\varphi'(x, y, t)$ still defines a well-ordering of $\mathbb{R}^{V^{\text{Random}_\lambda}}$. This is impossible by Lemma 1.4 because $\lambda \geq \omega_1$. $\square$

In particular, no projective (i.e., second-order in the language of arithmetic) formula defines a well-ordering of the reals, if the continuum is real-valued measurable. We close this Section with a fact that (we hope) helps to understand the form taken by the well-orderings obtained in the following sections.

The point is that we want to codify definability computations in the language of set theory within the language of third-order arithmetic.

Fact 3.32. *Let $\varphi(\vec{x})$ be a Σ^2_1-formula. Then, there is ψ, and a transitive structure $M \models \mathsf{ZFC}^{-\varepsilon}$ such that $\mathbb{R} \subseteq M$, $|M| = \mathfrak{c}$, or even ${}^\omega M \subseteq M$, such that for all reals $\vec{r}$,*

$$\varphi(\vec{r}) \iff M \models \psi(\vec{r}).$$

Here, $\mathsf{ZFC}^{-\varepsilon}$ is a sufficiently strong fragment of ZFC.

Proof: The existence of such an M is easily seen to be equivalent to a Σ^2_1-formula. Conversely, given φ, let η be large enough, so for any $\vec{r}$,

$$\varphi(\vec{r}) \iff V_\eta \models (\mathcal{P}(\mathbb{R}), \mathbb{R}, \omega, \dots) \models \varphi(\vec{r}),$$

and we can take as M a suitable substructure of V_η. $\square$

We have stated the fact in an informal manner, to emphasize its flexibility. For a specific version, we can take $\mathsf{ZFC}^{-\varepsilon}$ to mean, in this case, $\mathsf{ZFC}^- + \mathcal{P}(\mathbb{R})$ exists (considering a large H_η instead of V_η), or $\mathsf{ZFC}\restriction_{\Sigma_{200}}$, i.e., ZFC with replacement restricted to Σ_{200}-statements.

Remark 3.33. In fact, the pointclass Σ^2_1 can be identified by this method with the class $\Sigma_1(H_{\mathfrak{c}^+}, \in, H_{\omega_1})$, where H_{ω_1} is seen as a parameter and therefore quantification over it is considered bounded. This identification propagates to the classes Σ^2_n and $\Sigma_n(H_{\mathfrak{c}^+}, \in, H_{\omega_1})$ for $n < \omega$.

The fact and this remark motivate the general structure of the constructions that produce Σ^2_n-well-orderings: A model needs to be produced satisfying certain first order property ψ (somehow related to properties of the surrounding universe). Since the model can resemble the first-order theory of the surrounding universe as much as necessary, the need to satisfy ψ is in general not the problem. The difficulty arises in trying to isolate the model or models that we have in mind from possibly fake ones, which can be thought of as proving a "correctness" theorem. This general framework will be illustrated with the results of the following sections.

3.2 Woodin's construction

We begin with a construction in essence due to Woodin, which starting with a measurable cardinal produces a model where the cardinal is real-valued measurable, and the generic codes a subset of the reals. This construction is the prototype of the arguments to come later on. Working over $L[\mu]$, Woodin uses it to show the consistency of real-valued measurability of the continuum, together with a Δ^2_1-well-ordering of $\mathbb{R}$. We obtain the consistency of $\mathsf{RVM}(\mathfrak{c})$ together with a Δ^2_2-well-ordering of $\mathbb{R}$ without any restrictions in the large cardinal structure of the universe.

Start with $V \models \kappa$ $\texttt{is measurable and}$ $2^\kappa = \kappa^+$, and let $j : V \to N$ be a witnessing ultrapower embedding, so $j = j_{\mathcal{U}}$ for $\mathcal{U}$ a normal measure on κ. Let $\mathbb{Q} = \texttt{Random}_\kappa$ and $\mathbb{P}$ be the Easton product over the inaccessibles $\lambda < \kappa$ of $\texttt{Add}(\lambda^+, 1) \times \texttt{Add}(\lambda^{++}, 1)$.

Force over V with $\mathbb{P} \times \mathbb{Q}$, and let $G_\mathbb{P} \times G_\mathbb{Q}$ be generic.

Claim 3.34. *$\mathbb{P}$ preserves the measurability of κ. In fact, there is $G^* \in V$ such that whenever G is $\mathbb{P}$-generic over V, $G \times G^*$ is $j(\mathbb{P})$-generic over N, and we can lift j to an embedding $j_1 : V[G] \to N[G \times G^*]$.*

(*Cf.* [H], Lemma 2.2.4 or [C], Fact 3.1)

Proof: By elementarity, in N, $j(\mathbb{P}) = \mathbb{P} \times \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the Easton product of $\texttt{Add}(\lambda^+, 1) \times \texttt{Add}(\lambda^{++}, 1)$ over the inaccessibles $\lambda \in [\kappa, j(\kappa))$. In N, this set is κ^+-closed. But ${}^\kappa N \subset N$, so in fact it is κ^+-closed in V. Now notice that $|\mathcal{P}^N(\mathbb{P}_{\text{tail}})| = |(2^{j(\kappa)})^N| = |j(2^\kappa)| \leq (2^\kappa)^\kappa = 2^\kappa = \kappa^+$, where the last equality holds by hypothesis. Thus, the number of dense subsets of $\mathbb{P}_{\text{tail}}$ which belong to N is at most κ^+, and a straightforward induction lets us build (in V) a decreasing sequence of conditions which meet all of them. The filter G^* they generate is therefore $\mathbb{P}_{\text{tail}}$-generic over N.

It remains to argue that if G is $\mathbb{P}$-generic over V, then $G \times G^*$ is $j(\mathbb{P})$-generic over N, which amounts to show G and G^* are mutually generic. If so, j lifts to j_1 in the usual way.

But this is clear: Since $N[G^*] \subseteq V$, if G is $\mathbb{P}$-generic over V, it is also $\mathbb{P}$-generic over $N[G^*]$. $\triangle$

Notice that $\mathbb{P}$ is ω_1-closed, so $\mathbb{R}^{V[G_\mathbb{P}\times G_\mathbb{Q}]} = \mathbb{R}^{V[G_\mathbb{Q}]}$. In $V[G_\mathbb{Q}]$, let $A \subset \kappa$ code a well-ordering of $\mathbb{R}$ in order type κ.

Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ be the increasing enumeration of the inaccessibles in V below κ. For $\alpha < \kappa$, let G_α be the α^{th} component of $G_\mathbb{P}$, so G_α is the product of an $\mathtt{Add}(\delta_\alpha^+, 1)$-generic and an $\mathtt{Add}(\delta_\alpha^{++}, 1)$-generic over V. Let G_α^* be the $\mathtt{Add}(\delta_\alpha^+, 1)$-generic, if $\alpha \in A$, and the $\mathtt{Add}(\delta_\alpha^{++}, 1)$-generic, if $\alpha \notin A$. Finally, let

$$g = \prod_{\alpha<\kappa} G_\alpha^*.$$

Notice A is definable from g.

Claim 3.35. *$\kappa = \mathfrak{c}$ stays real-valued measurable in $V[G_\mathbb{Q}][g]$.*

Proof: By Theorem 3.30, $j(\mathbb{Q})/\mathbb{Q}$ is isomorphic to $\mathtt{Random}_{j(\kappa)}$ in $V[G_\mathbb{Q}]$. It follows that in $V[G_\mathbb{Q}][G_\mathbb{P}]$ as well as in $V[G_\mathbb{Q}][g]$, $j(\mathbb{Q})/\mathbb{Q}$ is still a complete measure algebra, since the forcing for which g is generic is a factor of $\mathbb{P}$, which is ω_2-closed in V and therefore ω_2-dense in $V[G_\mathbb{Q}]$ by Easton's Lemma, see [C], Fact 4.1. Since $j(\mathbb{Q})/\mathbb{Q}$ was homogeneous in $V[G_\mathbb{Q}]$, it is still homogeneous in $V[G_\mathbb{Q}][g]$ and in $V[G_\mathbb{Q}][G_\mathbb{P}]$. We conclude that $j(\mathbb{Q})/\mathbb{Q}$ is still isomorphic to $\mathtt{Random}_{j(\kappa)}$, by Maharam's theorem.

Let H be $j(\mathbb{Q})/\mathbb{Q}$-generic over $V[G_\mathbb{Q}][G_\mathbb{P}]$. We will show that in $V[G_\mathbb{Q}][g][H]$, j lifts to

$$j^* : V[G_\mathbb{Q}][g] \to N[j^*(G_\mathbb{Q})][j^*(g)].$$

This amounts to define $j^*(G_\mathbb{Q})$ and $j^*(g)$, and to check that the induced map j^* is well-defined and elementary. Once this is done, Solovay's theorem implies the claim.

Set $j^*(G_\mathbb{Q}) = G_\mathbb{Q}{}^\frown H$. To define $j^*(g)$, it suffices to define $j^*(g)_{[\kappa,j(\kappa))}$ (so $j^*(g) = g^\frown j^*(g)_{[\kappa,j(\kappa))}$). The intention is that the definition of $j^*(g)$ copies that of g, so we must start by defining $j^*(A)$.

Subclaim 3.36. *In $V[G_\mathbb{P}][G_\mathbb{Q}][H]$, j_1 lifts to $j_2 : V[G_\mathbb{P}][G_\mathbb{Q}] \to N[G_\mathbb{P}][G^*][G_\mathbb{Q}][H]$. The restriction of j_2 to $V[G_\mathbb{Q}]$ is an embedding*

$$j_3 : V[G_\mathbb{Q}] \to N[G_\mathbb{Q}][H]$$

definable in $V[G_\mathbb{Q}][H]$.

Proof: This is the usual way of showing that if ρ is measurable, then it is still real-valued measurable in $V^{\mathtt{Random}_\rho}$. As expected, simply set

$$j_2(\tau_{G_\mathbb{Q}}) = j_1(\tau)_{G_\mathbb{Q} \frown H},$$

for τ a $\mathbb{Q}$-name in $V[G_\mathbb{P}]$. The standard arguments (see [C], Fact 2.1.) prove j_2 is well-defined and elementary. Since j_1 extends j, $j_3 = j_2 \restriction_{V[G_\mathbb{Q}]} : V[G_\mathbb{Q}] \to N[G_\mathbb{Q}][H]$ is given by $j_3(\tau_{G_\mathbb{Q}}) = j(\tau)_{G_\mathbb{Q} \frown H}$ for τ a $\mathbb{Q}$-name in V, and is definable in $V[G_\mathbb{Q}][H]$ as claimed. ∇

Since $A \in V[G_\mathbb{Q}]$, $j_3(A) \in V[G_\mathbb{Q}][H]$. We set $j^*(A) = j_3(A)$. The key observation is that we do not really need a whole $j(\mathbb{P})$-generic to define $j^*(g)_{[\kappa, j(\kappa))}$, but a $\mathbb{P}_{\text{tail}}$-generic suffices: Remember that G^*, as built in Claim 3.34, is in V. We can now set

$$j^*(g)_{[\kappa,j(\kappa))} := \prod_{\alpha \in [\kappa, j(\kappa))} G^{**}_\alpha,$$

where G^{**}_α is the $\mathtt{Add}(\delta^+_\alpha, 1)$-generic added by G^* to N, if $\alpha \in j^*(A)$, and the $\mathtt{Add}(\delta^{++}_\alpha, 1)$-generic, if $\alpha \notin j^*(A)$. Here, $\langle \delta_\alpha : \alpha < j(\kappa) \rangle = j(\langle \delta_\alpha : \alpha < \kappa \rangle)$ is the increasing enumeration of the inaccessibles in N below $j(\kappa)$.

Extend j^* to a map

$$j^* : V[G_\mathbb{Q}][g] \to N[j^*(G_\mathbb{Q})][j^*(g)]$$

in the usual way. Notice that j^* is simply the restriction of j_2, as defined in Subclaim 3.36, to $V[G_\mathbb{Q}][g]$. This proves j^* is well-defined and elementary. Finally, notice that j^* is definable in $V[G_\mathbb{Q}][g][H]$. This concludes the proof. $\triangle$

Remark 3.37. The argument just given is quite general. It works as long as $\mathbb{P}$ is a reasonably definable product of sufficiently closed small forcings. The set we called A can code any subset of the reals in $V[G_\mathbb{Q}][G_\mathbb{P}]$. By coding A inside a "subproduct" g of $G_\mathbb{P}$, we avoid having to set up any sort of book-keeping devices in the ground model in order to define the well-ordering alongside the iteration. As a matter of fact, we do not need to worry about defining in the ground model (as an iteration or otherwise) the forcing whose generic is g.

Notice also that, in spite of this generality, some argument was required, since it is not necessarily true that if W is a forcing extension of V preserving $\mathsf{RVM}(\kappa)$, then any intermediate extension $V \subseteq M \subseteq W$ satisfies $\mathsf{RVM}(\kappa)$ as well. This observation is folklore and most likely due to Kunen.

We will show how to obtain a Σ^2_1-well-ordering of $\mathbb{R}$ using the forcing just described, as long as $V = L[\mu]$. A variation of the argument of this Section provides us with a Σ^2_2-well-ordering, without imposing any large cardinal restrictions on V. We give this argument first.

3.3 Σ^2_2-Well-orderings

The construction from last Section is quite flexible. As a matter of illustration, let us show how a variant of it produces a model with a Δ^2_2-well-ordering of $\mathbb{R}$ and $\mathsf{RVM}(\mathfrak{c})$. The combinatorial tool we use to carry out our coding was first considered in [ASh1], in the presence of MA.

Theorem 3.38. *If κ is measurable in V and $2^\kappa = \kappa^+$, then there is a forcing $\mathbb{F}$ of size κ such that*

$$1 \Vdash_{\mathbb{F}} \mathfrak{c} = \kappa \texttt{ is real-valued measurable, and there is a } \Delta^2_2 \texttt{ well-ordering of } \mathbb{R}.$$

Proof: By a preliminary forcing, if necessary, we may assume GCH holds below κ.

Again, let $\mathbb{Q} = \texttt{Random}_\kappa$. Let $\mathbb{P}$ be the Easton product over inaccessibles $\lambda < \kappa$ of $\prod_{n\in\omega} \texttt{Add}(\lambda^{+1+3n}, \lambda^{+3+3n})$, where the product is inverse. Let $\mathbb{S} = \mathbb{P} \times \mathbb{Q}$, and let $G_\mathbb{P} \times G_\mathbb{Q}$ be $\mathbb{S}$-generic over V.

As before:

- If $j : V \to N$ is an ultrapower embedding by a normal measure on κ, then j lifts to $j : V[G_\mathbb{P}] \to N[G_\mathbb{P}][G^*]$, where if $j(\mathbb{P}) = \mathbb{P} \times \mathbb{P}_{\text{tail}}$, then $G^* \in V$ is $\mathbb{P}_{\text{tail}}$-generic over N.
- $j(\mathbb{Q})/\mathbb{Q}$ is isomorphic to an appropriate random forcing in any intermediate model between $V[G_\mathbb{Q}]$ and $V_1 := V[G_\mathbb{Q}][G_\mathbb{P}]$, inclusive, and $\mathfrak{c} = \kappa$ is real-valued measurable in V_1. In fact if H is $j(\mathbb{Q})/\mathbb{Q}$-generic over V_1 then, in $V_1[H]$, j lifts to $j : V_1 \to N[G_\mathbb{P}][G^*][G_\mathbb{Q}][H]$, thus showing $\mathsf{RVM}(\mathfrak{c})$ in V_1, by Solovay's theorem.
- Similarly, in $V[G_\mathbb{Q}][H]$, j lifts to $j : V[G_\mathbb{Q}] \to N[G_\mathbb{Q}][H]$.
- $\mathbb{R}^{V[G_\mathbb{Q}]} = \mathbb{R}^{V[G_\mathbb{Q}][G_\mathbb{P}]}$.

In $V[G_\mathbb{Q}]$, let $A = \langle r_\alpha : \alpha < \kappa \rangle$ be a well-ordering of $\mathbb{R}$. In $V[G_\mathbb{Q}][G_\mathbb{P}]$, define g as follows:

Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ enumerate in V the inaccessibles below κ. Let G_α be the part of $G_{\mathbb{P}}$ which is generic for $\prod_{n\in\omega} \mathtt{Add}(\delta_\alpha^{+1+3n}, \delta_\alpha^{+3+3n})$. Write $G_\alpha \cong \prod_{n\in\omega} G_\alpha(n)$, where $G_\alpha(n)$ is the part of G_α generic for $\mathtt{Add}(\delta_\alpha^{+1+3n}, \delta_\alpha^{+3+3n})$. Then

$$g = \prod_{\alpha<\kappa} G_\alpha^*,$$

where $G_\alpha^* = \prod_{n\in\omega} G_\alpha^*(n)$ and

$$G_\alpha^*(n) = \begin{cases} G_\alpha(n) & \text{if } n \in r_\alpha, \\ 1_{\mathtt{Add}(\delta_\alpha^{+1+3n}, \delta_\alpha^{+3+3n})} & \text{if } n \notin r_\alpha. \end{cases}$$

The same argument as in Claim 3.35 shows G^* and $j(A)$ suffice to define $j^*(g)$ (and recall $G^* \in V$ and $j(A) \in V[G_{\mathbb{Q}}][H]$). It follows as in that Claim that $\mathfrak{c} = \kappa$ is real-valued measurable in $V[G_{\mathbb{Q}}][g]$, and that a lifting of j to $j^* : V[G_{\mathbb{Q}}][g] \to N[G_{\mathbb{Q}}][H][j^*(g)]$ definable in $V[g][G_{\mathbb{Q}}][H]$ serves as a witness.

All what remains is to see that we can "decode" the well-ordering A from g in a Σ^2_2-way in $V[G_{\mathbb{Q}}][g]$. The forcing $\mathbb{F}$ is then the factor of $\mathbb{S}$ for which $G_{\mathbb{Q}} \times g$ is a generic.

The key to our coding is the following notion (see [ASh2]):

Definition 3.39. Let λ be regular. The *club base number* for λ is

$$\min\{\, X \subseteq \mathcal{P}(\lambda) : \forall \text{ club } C \subseteq \lambda\, \exists \text{ club } D \in X\, (D \subseteq C) \,\}.$$

So the club base number for λ is the coinitiality of the club filter at λ, ordered under inclusion. Any collection X of clubs in λ realizing the minimum above generates the club filter at λ by closing under supersets.

If λ is regular and $2^\lambda = \lambda^+$, then the club base number for λ is λ^+, while if λ^{++} Cohen subsets of λ are added, their closures are clubs containing no club from the ground model, and mutual genericity guarantees that the club base number at λ is λ^{++}.

It follows that in $V[g]$ the inaccessibles below κ are just the δ_α, $\alpha < \kappa$, and the club base number for δ_α^{+1+3n} is either δ_α^{+2+3n} or $\delta_\alpha^{+3(n+1)}$ depending on whether $G_\alpha^*(n)$ is trivial or not, since the base number for $\delta_{\alpha_1}^{+1+3n}$ is not affected by forcing with (a subproduct of) $\prod_{m\in\omega} \mathtt{Add}(\delta_{\alpha_2}^{+1+3m}, \delta_{\alpha_2}^{+3+3m})$ for $\alpha_2 \neq \alpha_1$.

Maybe a more detailed argument is in order: Let $\lambda < \kappa$ be inaccessible, let $n < \omega$, and write $\mathbb{P} \cong \mathbb{P}_{\lambda,n} \times \mathtt{Add}(\lambda^{+1+3n}, \lambda^{+3+3n}) \times \mathbb{P}^{\lambda,n}$, where $\mathbb{P}_{\lambda,n}$ corresponds to the factors of $\mathbb{P}$ that add Cohen subsets to cardinals strictly smaller than λ^{+1+3n}, and $\mathbb{P}^{\lambda,n}$ corresponds to those factors that add Cohen subsets to strictly bigger cardinals. Then $\mathbb{P}^{\lambda,n}$ is sufficiently closed that it cannot ("by accident") add a subset of λ^{+1+3n}, while $\mathbb{P}_{\lambda,n}$ satisfies a sufficiently small chain condition that any club subset of λ^{+1+3n} that it adds contains a club in the ground model. It follows that in $V[G_{\mathbb{P}}]$, and therefore in $V[g]$, the only club base numbers that are affected are those that we have explicitly changed by means of g.

Finally, $G_{\mathbb{Q}}$ is added by ccc forcing, so it does not affect any of the club base numbers that concern us.

Now observe that in $V[G_{\mathbb{Q}}][g]$ we can define A, or rather the corresponding order relation $<_A$ on $\mathbb{R}$ as follows:

Let $\Psi(M)$ denote the conjunction of the following requirements:

> $M \models \mathsf{ZFC}^{-\varepsilon}$, M is transitive, $|M| = \mathfrak{c}$, and $\mathbb{R} \subseteq M$.[9] Moreover,
>
> 1. M computes cofinalities correctly, that is, if $\lambda, \mu \in M$, and there is $f : \lambda \to \mu$ cofinal, then there is such an $f \in M$.
> 2. For all $C \subseteq \lambda < \mathfrak{c}$ club, there is $D \in M$, $D \subseteq \lambda$ club, such that $D \subseteq C$.
> 3. M computes club base numbers correctly, that is, for all $\lambda < \mathfrak{c}$, $\mathcal{F} \subseteq \mathcal{P}(\lambda)^M$ collection of clubs, $|\mathcal{F}| < \mathfrak{c}$, there is $\mathcal{G} \in M$ collection of clubs such that $\mathcal{G}$ is coinitial in $\mathcal{F}$.[10]
>
> Finally, for all $r \in \mathbb{R}$ there is in M a unique sequence of club base numbers starting at a weakly inaccessible[11] which (in the obvious way) code r.

Notice that $\Psi(M)$ is a $\Pi_1(H_{\mathfrak{c}^+}, \in, H_{\omega_1})$-statement about M and that M does not have any cardinals above $\mathfrak{c}$.

For x, y reals, let $\psi(x, y)$ hold iff

> There is M such that $\Psi(M)$ holds and in M the sequence coding x appears before the sequence coding y.

Here we can take $\mathsf{ZFC}^{-\varepsilon}$ to mean ZFC^-, in which case $\mathcal{P}(\lambda)^M$, as in 3. above, is to be interpreted as a definable class. This does not affect the desired complexity of ψ.

The relation ψ just defined can be rendered Σ^2_2 in a straightforward fashion. We are done once we verify that $x <_A y$ holds for reals x, y if and only if $\psi(x, y)$ does. That $x <_A y$ implies $\psi(x, y)$ is easy, $M = V[G_{\mathbb{Q}}][g]_\kappa$ is a witness. To see the converse just observe

[9] So $\mathrm{ORD}^M \geq \mathfrak{c}$.

[10] The requirement on the size of $\mathcal{F}$ is not essential. We just include it to ensure the universal quantifier in the definition of the well-ordering we obtain actually ranges over bounded subsets of $\mathfrak{c}$.

[11] We do not introduce any "fake" codings this way, since the ground model satisfied GCH. The coding could have occurred at many other places (say, starting at limit cardinals), so this is by no means essential.

that any M witnessing $\psi(x, y)$ is correct about cofinalities below κ, and computes correctly club base numbers of cardinals below κ. The uniqueness of the coding of reals by club base numbers ensures that even if $\mathrm{ORD}^M > \mathfrak{c}$, no fake codings (witnessing false relations $x <_A y$) may arise. Since g was defined precisely to code A using the club base numbers, $\psi(x, y)$ implies $x <_A y$. This completes the proof. □

Let $(\Sigma^2_2)^+$ denote the class of statements about the reals expressible as a Boolean combination of Σ^2_2-statements. As a consequence of the argument above and Solovay's theorem on preservation of real-valued measurability (Fact 3.24) we obtain that generic invariance of $(\Sigma^2_2)^+$ with respect to real-valued measurability of the continuum[12] is not a theorem of ZFC. In effect, the statement "ψ defines a well-ordering of $\mathbb{R}$", for ψ as above, is a $(\Sigma^2_2)^+$-statement[13], it can be made true over V as long as there are measurable cardinals in V, and can be made false afterwards simply by adding ω_1 many Random reals, by Corollary 1.5.

3.4 Anticoding Results

Of course, the Δ^2_2 above is an overkill; notice the third-order universal quantifier only ranges over *bounded* subsets of κ. It is natural to wonder whether we can improve the complexity of the well-ordering to be Σ^2_1. The problem with following a strategy similar to the one just described is that we need to ensure correctness of the model M with respect to the combinatorial structure of the universe that carries out the coding (the club base numbers, for example). This level of correctness needs to be attained via projective statements. This seems to suggest that we need to be able to code (suitable) bounded subsets of κ by reals. In general (as in the arguments of [ASh1] and [ASh2], see also our proof of Theorem 2.1), this is done by arranging that the universe satisfies something like a sufficiently strong fragment of MA to be able to use the coding provided by almost-disjoint forcing.

[12] See after Theorem 1.21 for the definition.

[13] It is not quite Σ^2_2, even though the relation ψ is Δ^2_2: Let ψ_1 and ψ_2 be Σ^2_2-formulas such that for all reals r, s, $\psi(r, s) \Leftrightarrow \psi_1(r, s) \Leftrightarrow \neg\psi_2(r, s)$. Then either transitivity of ψ is not expressed in a Σ^2_2-way if, say, only ψ_1 is used to describe ψ, or else we need to include the clause that $\psi_1(x, y) \Leftrightarrow \neg\psi_2(x, y)$ holds for all reals x, y.

3.4.1 MA_{ω_1}

Unfortunately, MA itself fails after adding even one random real, so it is incompatible with real-valued measurability of the continuum.

Theorem 3.40. *If* $\mathsf{RVM}(\kappa)$ *holds and* $\kappa \leq \mathfrak{c}$*, then there is a ccc partial order* $\mathbb{P}$ *such that* $\mathbb{P} \times \mathbb{P}$ *is not ccc.*

Corollary 3.41. *If* $\mathsf{RVM}(\kappa)$ *holds and* $\kappa \leq \mathfrak{c}$*, then* MA_{ω_1} *fails.* □

The hypothesis we display is not ideal, but there is some subtlety here, since Prikry showed that MA is compatible with quasi-measurability of the continuum, see [F1], Proposition 9G. Theorem 3.40 was known to Fremlin.

Remark 3.42. Corollary 3.41 was known independently of the theorem. Arguments more in the spirit of forcing axioms are possible: For example, if κ is atomlessly measurable, then

- $\mathrm{non}(\mathbb{R}, \mathcal{N}) = \mathrm{cov}(\mathbb{R}, \mathcal{M}) = \mathrm{add}(\mathcal{M}) = \mathrm{add}(\mathcal{N}) = \mathfrak{p} = \omega_1$. Here, $\mathcal{N}$ is the ideal of Lebesgue null sets and $\mathcal{M}$ is the ideal of meager sets.
- $\mathfrak{b} < \kappa$.

See [F1] and references within. The particular case $\mathsf{RVM}(\mathfrak{c}) \Rightarrow \mathfrak{b} < \mathfrak{c}$ is due to Banach-Kuratowski ([BaKu]).

It is well known that $\mathfrak{p}$ is the smallest cardinal λ such that MA_λ(σ-centered) fails. Recall that almost disjoint forcing is σ-centered.

Proof of Theorem 3.40: This is a corollary of the following result of Roitman[14] ([R]):

Lemma 3.43. *In* $V^{\mathtt{Random}_\omega}$ *there is a ccc partial order whose square is not ccc.* △

Corollary 3.44. *Roitman's result 3.43 holds in* $V^{\mathtt{Random}_\lambda}$ *and not just* $V^{\mathtt{Random}_\omega}$.

Proof: Since $\mathtt{Random}_\lambda/\mathtt{Random}_\omega \cong \mathtt{Random}_\lambda$, this follows from Corollary 3.27. △

Assume $\mathsf{RVM}(\kappa)$ where $\kappa \leq \mathfrak{c}$, and let λ be such that in $V^{\mathtt{Random}_\lambda}$ there is an embedding $j : V \to N$ with $\mathrm{cp}(j) = \kappa$. By the Corollary there is in $V^{\mathtt{Random}_\lambda}$ a ccc partial

[14] In [BJ] Theorem 3.2.30, this is erroneously attributed to Galvin. Galvin devised a general method to construct such posets. Roitman showed that the construction works in $V^{\mathbb{F}}$, where $\mathbb{F} = \mathtt{Add}(\omega, 1)$ or $F = \mathtt{Random}_\omega$.

order $\mathbb{P}$ whose square is not ccc. By taking Skolem hulls, we may assume $|\mathbb{P}| \leq \aleph_1$. Since N is closed under ω_1-sequences, we may assume $\mathbb{P} \in N$ and

$$N \models \mathbb{P} \texttt{ is ccc but } \mathbb{P} \times \mathbb{P} \texttt{ is not.}$$

But then, by elementarity, there is such a partial order in V. $\square$

Question 3.45 (Fremlin). *Suppose κ is atomlessly measurable. Are there two ccc posets $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ has an antichain of size κ?*

3.4.2 OCA

Another forcing axiom that is used to code information about subsets of reals is the Open Coloring Axiom OCA.

Recall:

Definition 3.46. 1. Let X be a topological space. Recall that $[X]^2 := \{ \{x, y\} \subseteq X : x \neq y \}$. Let Δ_X denote the diagonal of $X \times X$, so $\Delta_X = \{ (x, x) : x \in X \}$. For $A \subseteq [X]^2$, let $\tilde{A} = \{ (x, y) : \{x, y\} \in A \}$. An *open coloring* of $[X]^2$ is a partition $[X]^2 = K_0 \dot\cup K_1$ such that $\tilde{K}_0$ is open in $X^2 \setminus \Delta_X$. $Y \subseteq X$ is *i-homogeneous*, for $i \in 2$ (with respect to a given coloring), iff $[Y]^2 \subseteq K_i$.

2. The Open Coloring Axiom OCA holds iff for all separable metric spaces X, for all open colorings $[X]^2 = K_0 \dot\cup K_1$ it is either the case that

   (a) There is an uncountable 0-homogeneous $Y \subseteq X$, or

   (b) X is the union of countably many 1-homogeneous sets.

For information on OCA see for example [Fu].

The reason why OCA is considered a forcing axiom is the following result:

Definition 3.47. Let S be a set and let $[S]^{<\omega} = K_0 \dot\cup K_1$ be a partition of the finite subsets of S into two classes. The partition $\{K_0, K_1\}$ is *ccc-destructible* iff there is a ccc forcing $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{X}$ for a 0-homogeneous set:

$$1 \Vdash_{\mathbb{P}} [\dot{X}]^{<\omega} \subseteq K_0,$$

such that any $s \in S$ is forced by some condition to be in $\dot{X}$.

Theorem 3.48 (Todorčević). MA *holds iff whenever S is an uncountable set of size $< \mathfrak{c}$ and $[S]^{<\omega} = K_0 \dot\cup K_1$ is a ccc-destructible partition, then S can be covered by countably many 0-homogeneous sets.* □

Theorem 3.49. *If* RVM(κ) *holds and $\kappa \leq \mathfrak{c}$, then* OCA *fails.*

This is essentially due to Todorčević.

Proof: The key to this result is the notion of an entangled linear order.

Definition 3.50. Let $(X, \leq)$ be a linear order. A set $A \subseteq X$ is ω_1*-entangled* iff for every $n < \omega$ and for every double-sequence $\langle x_{\xi,m} : \xi < \omega_1, m < n \rangle$ of pairwise distinct elements of A, for every $u \subseteq n$ there are $\xi^1 < \xi^2 < \omega_1$ such that $u = \{ m \in n : x_{\xi^1,m} < x_{\xi^2,m} \}$.

I think the following is due to Todorčević and Baumgartner, see [Fu] for a proof.

Fact 3.51. *If there is an uncountable ω_1-entangled subset of $\mathbb{R}$, then* OCA *fails*[15]. △

We are done, by [F1] Proposition 7F:

Lemma 3.52 (Todorčević). *If κ is atomlessly measurable, then for every $\lambda < \kappa$ there is an ω_1-entangled subset of $\mathbb{R}$ of size λ.* △

Actually, what Todorčević claims ([To1] Theorem 2) is the following folklore result:

Lemma 3.53. *If E is a set of Random reals, then E is $\aleph_1$-entangled.* △

In fact, something slightly better holds. The corresponding result for $V^{\mathtt{Add}(\omega,1)}$ is due to Yuasa, see [Y], Theorem 2.2 Claim (2), and can be adapted to give 3.54. I imagine this result must be folklore by now, but I have been unable to find an explicit reference.

Lemma 3.54. $V^{\mathtt{Random}_\omega} \models$ `There is an uncountable` ω_1`-entangled subset of` $\mathbb{R}$.
△

Observe that $\mathtt{Random}_\lambda \cong \mathtt{Random}_\lambda * \dot{\mathtt{Random}}_\omega$, by Maharam's theorem. Theorem 3.49 now follows as before: For some λ, in $V^{\mathtt{Random}_\lambda}$ there is an embedding $j : V \to N$ with $\mathrm{cp}(j) = \kappa$ and ${}^{\omega_1}N \subseteq N$, so in N there is an uncountable ω_1-entangled subset of $\mathbb{R}$ and, by elementarity, there is such a set also in V. By 3.51, OCA fails in V.

[15]This also contradicts MA_{ω_1}, thus giving yet another proof of Corollary 3.41.

Of course, no additional appeal to Maharam's theorem is required if instead of Lemma 3.54, Todorčević Lemma 3.53 is used. □

These arguments should make it clear that any statement sufficiently fragile in the sense that random forcing destroys it and sufficiently absolute in the sense that it transfers to the generic ultrapower of the ground model, is bound to fail if there are atomless measurable cardinals. Thus, any naive attempt to solve the problem we were facing at the end of last Section, namely, the coding of bounded subsets of κ by reals (where κ was measurable in the ground model and turns atomlessly measurable in the extension), say by including into the product we were calling $\mathbb{P}$ small factors that will do the coding of bounded subsets, runs into the immediate difficulty that we are adding random reals by homogeneous forcing (by the poset we were calling $\mathbb{Q}$, which is just $\mathtt{Random}_\kappa$), which most likely will undo our coding. We would have to do the coding in such a way that no initial segment of the iteration would suffice, but this seems difficult as well, because bounded sets of κ would most likely appear in initial segments of the iteration.

3.4.3 Real-valued huge cardinals

A Σ^2_2-well-ordering, on the other hand, can not be obtained for free.

Definition 3.55. A cardinal κ is *real-valued huge* iff there is $\lambda \geq \omega$ such that in $V^{\mathtt{Random}_\lambda}$ there exists an elementary embedding $j : V \overset{\prec}{\longrightarrow} N$ with $\mathrm{cp}(j) = \kappa$ and such that ${}^{j(\kappa)}N \subseteq N$.

The following is clear:

Lemma 3.56. *If κ is huge, then* $V^{\mathtt{Random}_\kappa} \models \kappa = \mathfrak{c}$ `is real-valued huge`.

Proof: Let $j : V \to M$ in V witness hugeness of κ, so ${}^{j(\kappa)}M \subseteq M$ and $\mathrm{cp}(j) = \kappa$. Set $\mathbb{Q} = \mathtt{Random}_\kappa$. Let G be $\mathbb{Q}$-generic over V, and let H be $j(\mathbb{Q})/\mathbb{Q}$-generic over $V[G]$. We just need to verify that in $V[G^\frown H]$, j lifts to

$$j^* : V[G] \to M[G][H]$$

and that $V[G^\frown H] \models {}^{j(\kappa)}M[G][H] \subseteq M[G][H]$. As usual, the lifting j^* is given by $j^*(\tau_G) = j(\tau)_{G^\frown H}$. This is well-defined and elementary.

Given a sequence of names $\vec{\tau} = \langle \tau_\alpha : \alpha < j(\kappa) \rangle$ with each τ a $j(\mathbb{Q})$-name in M, the whole sequence $\vec{\tau}$ belongs to $M[G]$, by the ccc of $\mathbb{Q}$, and therefore $\langle (\tau_\alpha)_{G^\frown H} : \alpha < j(\kappa) \rangle \in M[G][H]$. From this the result follows. □

Having shown the consistency of real-valued hugeness of the continuum, we now point out the following observation due to Woodin:

Fact 3.57 (Woodin). *Suppose $\mathfrak{c}$ is real-valued huge. Then there are no third-order definable well-orderings of the reals.*

Proof: The same argument as for $L(\mathbb{R})$ in Theorem 3.31 works:

Towards contradiction, let $\varphi(x, y, z)$ be a third order formula in the language of arithmetic, and let $t \in \mathbb{R}$ be such that for some well-ordering $<$ of $\mathbb{R}$, $\varphi(r, s, t)$ holds of reals r, s iff $r < s$.

Let $\lambda \geq \omega$ and G a $\mathtt{Random}_\lambda$-generic over V be such that in $V[G]$ there is an embedding $j : V \to N$ with $\mathrm{cp}(j) = \mathfrak{c}^V$ and ${}^{j(\mathfrak{c}^V)}N \subseteq N$. Then $\mathcal{P}(\mathbb{R})^{V[G]} \subseteq N$, since $|\mathbb{R}| = j(\mathfrak{c}^V)$ holds in N (and $\mathbb{R}^N = \mathbb{R}^{V[G]}$, since ${}^\omega N \subseteq N$.) But this means that third order statements in the language of arithmetic, with parameters from N, are absolute between N and $V[G]$.

We are done, because by elementarity $\varphi(\cdot, \cdot, t)$ would be a third-order definition of a well-ordering of the reals in $V[G]$, but this is impossible by the Corollary to Lemma 1.4. $\square$

Remark 3.58. Notice that what the proof actually shows is that if $\mathfrak{c}$ is real-valued huge and λ is as in Definition 3.55, then $V \equiv_{\underset{\sim}{\Sigma}^2_\omega} V^{\mathtt{Random}_\lambda}$, where boldface indicates that real parameters from V are allowed.

The argument of Theorem 3.38 breaks down very early when trying to adapt it to the case where κ is huge. For example, the existence of the N-generic object we called G^* cannot be ensured due to the strong closure of N.

Remark 3.58 suggests the natural question of whether generic invariance of Σ^2_ω with respect to "$\mathfrak{c}$ is real-valued huge" holds. This seems somewhat delicate, since there does not seem to be a natural counterpart to Solovay's Fact 3.24 for preservation of real-valued hugeness. The hypothesis is by no means intended to be optimal. For example, it is not clear whether the natural real-valued version of $\mathcal{P}^2(\kappa)$-measurability of κ for $\kappa = \mathfrak{c}$ suffices to rule out the existence of third-order definable well-orderings of $\mathbb{R}$.

As expected, real-valued hugeness is a serious large cardinal assumption, strictly stronger than real-valued measurability. Here we content ourselves with some easy observations and a remark:

Fact 3.59. *If κ is real-valued huge, then there are weakly inaccessible cardinals larger than κ.*

Proof: Let λ be as in Definition 3.55, and in $V^{\texttt{Random}_\lambda}$, let $j : V \to N$ be the witnessing embedding. Then $N \models j(\kappa)$ `is real-valued measurable`, so in particular $N \models j(\kappa)$ `is weakly inaccessible`.

But $V[G] \models {}^{j(\kappa)}N \subseteq N$, so $j(\kappa)$ *is* weakly inaccessible in $V[G]$, and therefore in V.

As usual, the proof actually shows that there are fixed point of the weakly-Mahlo hierarchy, etc., above κ. □

Theorem 3.60. *If $\mathfrak{c}$ is real-valued huge, then the real-valued measurable cardinals are unbounded below $\mathfrak{c}$. In fact, for a witnessing probability ν, $\nu(\{\, \alpha < \mathfrak{c} : \mathsf{RVM}(\alpha) \,\}) = 1$.*

Proof: As before, let $\lambda \leq \omega$ be as in Definition 3.55. Let $\varphi : \texttt{Random}_\lambda \to [0,1]$ be the 'probability measure' associated to $\texttt{Random}_\lambda$, fix a $\texttt{Random}_\lambda$-generic G over V, and in $V[G]$ let $j : V \to N$ witness real-valued hugeness of $\mathfrak{c}^V$.

Let $\kappa = \mathfrak{c}^V$. Then $\mathsf{RVM}(\kappa)$ holds in $V[G]$, by Fact 3.24. Let $\hat{\nu} : \mathcal{P}(\kappa) \to [0,1]$ be a witness. Notice that $\mathcal{P}(\kappa) \in N$. Since $\mathfrak{c}^{V[G]} = j(\kappa)$ is inaccessible, and therefore $j(\kappa)^{2^\kappa} = j(\kappa)$, then in particular $\hat{\nu} \in N$. Thus, $N \models \mathsf{RVM}(\kappa)$.

Since G was arbitrary, $\varphi[\![\kappa \in \jmath(\{\, \alpha : \mathsf{RVM}(\alpha) \,\})]\!] = 1$, where $\jmath$ denotes a term for an embedding witnessing real-valued hugeness of $\mathfrak{c}$.

In V, let $\nu : \mathcal{P}(\kappa) \to [0,1]$ be defined as usual by $\nu(A) = \varphi[\![\kappa \in \jmath(A)]\!]$. Then ν is as required.

As usual, this proof actually gives that $\mathfrak{c}$ is limit of real-valued measurables that also concentrate on real-valued measurables that concentrate on real-valued measurables, etc. □

Remark 3.61. Real-valued huge cardinals imply the existence of inner models for Woodin cardinals. In the presence of measurable cardinals this is an immediate consequence of the following result of Steel, which appears in [St1], Theorem 7.1:

Theorem 3.62 (Steel). *Let Ω be measurable, and let G be $\mathbb{P}$-generic over V for some $\mathbb{P} \in V_\Omega$. Suppose that in $V[G]$ there is a transitive class M and an elementary embedding*

$$j : V \to M \subseteq V[G]$$

with $\mathrm{cp}(j) = \kappa$ *and such that* $V[G] \models {}^{<j(\kappa)}M \subseteq M$. *Then the* K^c*-construction reaches a non-1-small level*[16]. $\square$

In fact, much more follows from this hypothesis. For example, it is straightforward to improve the argument leading to Theorem 3.60 to a proof of the fact that there is a 'probability measure' $\nu : \mathcal{P}(\mathfrak{c}) \to [0,1]$ such that $\nu(\{\,\alpha : \alpha$ is real-valued almost huge$\,\}) = 1$. Here, a cardinal κ is called almost huge iff there is a $\lambda \geq \omega$ such that in $V^{\mathtt{Random}_\lambda}$ there is an embedding $j : V \to N$ with $\mathrm{cp}(j) = \kappa$ and such that $V^{\mathtt{Random}_\lambda} \models {}^{<j(\kappa)}N \subseteq N$.

By replacing the measurable Ω in Steel's result with sharps, and by a straightforward relativization, we obtain:

Corollary 3.63. *Suppose* $\mathfrak{c}$ *is real-valued huge. Then for all bounded* $A \subseteq \mathfrak{c}$, $M_1(A)^\sharp$ *exists.* $\square$

However, the strength of real-valued hugeness of $\mathfrak{c}$ lies well beyond 3.63. Using his technique of the core model induction, Woodin has recently obtained a significant improvement:

Theorem 3.64 (Woodin). *If there is a real-valued almost huge cardinal, then* $\mathsf{AD}^{L(\mathbb{R}^\sharp)}$ *holds*[17]. $\square$

Part of the argument leading to 3.64 is of a purely combinatorial nature, and we include it here.

Lemma 3.65 (Woodin). *If there is a real-valued measurable cardinal, then* $\mathbb{R}^\sharp$ *exists.*

Remark 3.66. Suppose $\mathsf{RVM}(\kappa)$. Of course we only need to consider the case where κ is atomlessly measurable. What makes 3.65 interesting is that we are not assuming $\kappa = \mathfrak{c}$. An easy argument shows that $\mathbb{R}^\sharp$ exists if $\mathsf{RVM}(\mathfrak{c})$ holds: All bounded subsets of $H_\mathfrak{c}$ admit sharps. Thus, if $\kappa = \mathfrak{c}^V$ and λ is such that $V^{\mathtt{Random}_\lambda} \models j : V \to N$ `witnesses real-valued measurability of` κ, `and` ${}^{\kappa}N \subseteq N$, then all bounded subsets of $j(\kappa)$ in N have sharps. Since ${}^{\kappa}N \subseteq N$, $\mathbb{R}^V \in N$ and is coded by such a bounded set. Hence, $N \models \mathbb{R}^\sharp$ `exists`, and therefore $V[G] \models \mathbb{R}^\sharp$ `exists`. Since set forcing cannot create sharps, $\mathbb{R}^\sharp \in V$. In fact, since $\mathtt{Random}_\lambda$ is homogeneous, finestructural versions of this fact follow as well. We proceed now with the proof of Lemma 3.65.

[16] I.e., $M_1^\sharp$, the sharp for a proper class finestructural inner model with a Woodin cardinal, exists.
[17] As usual, by $L(\mathbb{R}^\sharp)$ we mean $L(\mathbb{R} \cup \{\mathbb{R}^\sharp\})$.

Proof: Suppose $\kappa \leq \mathfrak{c}$ and $\mathsf{RVM}(\kappa)$. It suffices to show that $\Theta^{L(\mathbb{R})} < \kappa$, by [HaMaW], Theorem I.3.39, related to Theorem 1.57.[18] If $\Theta^{L(\mathbb{R})} < \kappa$, it follows from this theorem that $\mathbb{R}^\sharp$ exists in a forcing extension of V, and therefore in V.

Let j be a generic embedding witnessing $\mathsf{RVM}(\kappa)$, $j : V \to N$. j is defined in $V^{\mathrm{Random}_\lambda}$ for some $\lambda \geq \omega_1$, and $\mathbb{R}^N = \mathbb{R}^{V^{\mathrm{Random}_\lambda}}$. In $N^{(\mathrm{Random}_{j(\lambda)})^N}$, there is an embedding $i^* : L(\mathbb{R}^N) \to L(\mathbb{R}^{N^{(\mathrm{Random}_{j(\lambda)})^N}})$ with $i^* \restriction_{\mathrm{ORD}} = \mathrm{id}$, by Lemma 1.4 and Fact 3.20. Therefore, by elementarity of j, there is in $V^{\mathrm{Random}_\lambda}$ such an embedding $i : L(\mathbb{R}^V) \to L(\mathbb{R})^{V^{\mathrm{Random}_\lambda}}$.

Notice that from the existence of both embeddings j and i it follows that Θ must be a fixed point of j and in particular $\kappa \neq \Theta^{L(\mathbb{R})}$.

Towards a contradiction, suppose now that there is a surjection $\pi : \mathbb{R} \to \kappa$ with $\pi \in \mathsf{HOD}^{L(\mathbb{R})}$. Since i obviously fixes the reals of V, $j(\pi)\restriction_{\mathbb{R}^V}$ is ordinal definable in $L(\mathbb{R}^V)$, and is a surjection of $\mathbb{R}^V$ onto $j(\kappa)$. But then there is an $x \in \mathbb{R}^V$ such that $j(\pi)(x) = \kappa$, a contradiction: $j(\pi)(x) = j\big(\pi(x)\big) = \pi(x) < \kappa$. □

This Section has highlighted inherent difficulties that a proof of the consistency of $\mathsf{RVM}(\mathfrak{c})$ together with a Σ^2_1-well-ordering of the reals must face.

Woodin's result in the following Section solves them in an indirect manner, by restricting in a very serious way the universe over which the argument takes place. The question of whether measurability of κ and GCH (or for that matter, any set of hypotheses which do not carry anti-large cardinal restrictions, or smallness requirements on the universe) suffice to force a model of $\mathsf{RVM}(\mathfrak{c})$ with a Σ^2_1-well-ordering of $\mathbb{R}$ is still open. In Section 3.6 we discuss a possible alternative approach.

3.5 Σ^2_1-Well-orderings

Recall:

Definition 3.67. By $L[\mu]$ we mean the smallest proper class inner model of the theory

$$\mathsf{ZFC} + \text{“There exists a measurable,”}$$

in this context, by μ we always mean a witness to measurability, i.e.,

$$L[\mu] \models \mu \texttt{ is a normal } \kappa\texttt{-complete measure on some cardinal } \kappa.$$

[18] The reference is according to the version of [HaMaW] dated $9 \cdot \mathtt{xii} \cdot 1999$. The result states that $L(\mathbb{R})[H] = \mathsf{HOD}^{L(\mathbb{R})}[G]$. Here, H is a generic enumeration of $\mathbb{R}$ and G, defined in terms of H, is generic over $\mathsf{HOD}^{L(\mathbb{R})}$ for a poset in $V_\Theta^{\mathsf{HOD}^{L(\mathbb{R})}}$.

We abuse notation in the usual way, and occasionally talk about the theory $V = L[\mu]$. This should not add any confusion to anybody comfortable enough with the inherent tangle of syntax and semantics that characterizes set theory.

Theorem 3.68 (Woodin). *If $V = L[\mu]$, then Woodin's construction produces a model where $\mathsf{RVM}(\mathfrak{c})$ and there is a Δ^2_1-well-ordering of $\mathbb{R}$.*

Proof: We use the notation from Section 3.2. Recall: In $L[\mu]$, $\mathbb{Q} = \mathtt{Random}_\kappa$, where κ is measurable. $\mathbb{P}$ is the Easton product over the inaccessibles λ below κ of $\mathtt{Add}(\lambda^+, 1) \times \mathtt{Add}(\lambda^{++}, 1)$. $G_{\mathbb{P}} \times G_{\mathbb{Q}}$ is $\mathbb{P} \times \mathbb{Q}$-generic over $L[\mu]$. $A \subsetneq \kappa = \mathfrak{c}$ codes a well-ordering of $\mathbb{R}$, $A \in L[\mu][G_{\mathbb{Q}}]$. We may assume the order type of this well-ordering is κ. $g \subsetneq G_{\mathbb{P}}$ codes A as follows:

$$g = \prod_{\alpha<\kappa} G^*_\alpha,$$

where G^*_α is either the $\mathtt{Add}(\delta^+_\alpha, 1)$-generic or the $\mathtt{Add}(\delta^{++}_\alpha, 1)$-generic added by $G_{\mathbb{P}}$, depending on whether $\alpha \in A$ or not. Here, $\langle \delta_\alpha : \alpha < \kappa \rangle$ is the increasing enumeration of the inaccessibles of $L[\mu]$ below κ.

Let $V_1 = L[\mu][G_{\mathbb{Q}}][g]$, so $\mathbb{R}^{V_1} = \mathbb{R}^{L[\mu][G_{\mathbb{Q}}]}$. The main result of Section 3.2, Claim 3.35, is that $\mathsf{RVM}(\mathfrak{c})$ holds in V_1. We claim that the well-ordering coded by A is Σ^2_1 in V_1. This we verify by "guessing" the ground model. What the following claim formalizes is our intuition that any structure which resembles $L[\mu]$ sufficiently close must coincide with $L[\mu]$. This resemblance we indicate in terms of a covering property.

Definition 3.69. Let N be a transitive structure which models enough set theory. We say that N *satisfies countable covering* if and only if

$$\forall \sigma \in \mathcal{P}_{\omega_1}(N)\, \exists \tau \in N\, (\sigma \subseteq \tau \text{ and } N \models |\tau| \leq \aleph_0).$$

Claim 3.70. *In V_1, suppose M is transitive, $|M| = \mathfrak{c}$, $M \models \mathsf{ZFC}^{-\varepsilon} + V = L[\mu]$. Let κ_M be the measurable cardinal in the sense of M, and $\kappa = \mathfrak{c}$. Suppose $\kappa_M \geq \mathfrak{c}$, M is iterable and satisfies countable covering.*

Then $M_\kappa = L[\mu]_\kappa$.

The hypothesis of Claim 3.70 require some expansion. The point of the claim is that we have identified the ground model (or, better, the part of the ground model relevant to our argument) in a projective fashion. To be precise, take $\mathsf{ZFC}^{-\varepsilon}$ to mean $\mathsf{ZFC}\restriction_{\Sigma_{10^{10}}}$, ZFC with the replacement schema restricted to $\Sigma_{10^{10}}$-statements. Obviously, much less suffices.

See [Do] for a careful exposition of iterability at this level. What we refer to as K^{DJ} is just called K in [Do], and $L[\mu]$ is called there $L[U]$.

Proof: First notice that an initial segment of $L[\mu]$ itself satisfies the requirements: Iterability is clear, and countable covering holds because $\mathbb{Q}$ is ccc and $\mathbb{P}$ is ω_1-closed.

Assume M satisfies the requirements of the Claim. Suppose $M \not\supseteq L[\mu]_\kappa$, and let $a \in L[\mu]_\kappa \setminus M$.

Notice that (provably in $\mathsf{ZFC} +$ "$L[\mu]$ `exists`"), $K_\kappa^{DJ} = L[\mu]_\kappa$. Hence, there is a mouse $\bar{M} = L_{\bar{\alpha}}[\bar{\mu}]$ with $a \in \bar{M}_{\bar{\kappa}}$, where $\bar{\mu}$ is a $\bar{M}$-measure on $\bar{\kappa}$. Since $a \notin M$ and M does not move in the comparison of $\bar{M}$ with M, $\bar{M} \supseteq M$.

We must in fact have equality or the critical sequence (of $\bar{M}$) would violate covering since $\bar{\kappa} < \kappa$. When comparing $\bar{M}$ with M, M does not move so after iterating $\bar{M}$ ω-many times the critical points and their supremum stay below the measurable of M and must be inaccessibles of M as $\bar{M}$ iterates past M, but this clearly contradicts countable covering.

The other containment is clear. △

We are basically done now: To require iterability of a model M as in the Claim is a projective requirement (for example, if $M \models V = K^{DJ}$, iterability states that every countable mouse required to verify $V = K^{DJ}$ is iterable.) Hence, to define A in a Σ^2_1-way following the approach explained at the end of Section 3.1 it suffices to notice the following Claim, whose proof concludes the proof of the Theorem, and this Section.

Claim 3.71. *In V_1 suppose $\hat{\delta} < \kappa$ and $a \subseteq \hat{\delta}^+$ is such that $a \notin L[\mu]$ is $\mathtt{Add}(\hat{\delta}^+, 1)$-generic over $L[\mu]$. Then $\hat{\delta}$ is a limit or the successor of a limit cardinal δ_β, and $\beta \in A$ iff $\hat{\delta} = \delta_\beta$.*

It follows that A can be defined by refering to those cardinals $\hat{\delta}$ for which there is a set a as above.

Proof: This follows quite easily by what is essentially the decoding argument given during the proof of Theorem 3.38. △ □

Notice essentially the same argument provides models of a Σ^2_1-well-ordering together with $\mathsf{RVM}(\mathfrak{c})$, as long as the ground model is finestructural, and the iterability condition is projective[19]. Following this approach, granting large cardinals, and starting with a definable finestructural model, the forcing construction from Section 3.2 produces a

[19] If M is the model the corresponding version of Claim 3.70 tries to identify, a fake candidate would give rise to a club of inaccessibles below the distinguished measurable κ, again violating covering.

model of $\mathsf{RVM}(\mathfrak{c})$ together with a $\Sigma^2_1(\Gamma^\infty)$-well-ordering of $\mathbb{R}$. Here, $\Sigma^2_1(\Gamma^\infty)$ is the pointclass of sets of reals A such that for some projective formula ψ and some real parameter r, A can be defined by: For all $s \in \mathbb{R}$,

$$s \in A \iff \exists B\, (\psi(s, r, B) \text{ and } B \in \Gamma^\infty).$$

The pointclass Γ^∞ consists (under the background assumption that there are unboundedly many Woodin cardinals) of all Universally Baire sets of reals.

The notation here is a bit tricky, and somewhat unfortunate.

Definition 3.72 **(Γ^κ).** For κ a cardinal, by Γ^κ we denote the pointclass of κ-Homogeneous sets of reals.

A set $A \subseteq \mathbb{R}^n$ is κ-Homogeneous iff $A = p[T]$ for some tree T admitting a homogeneity system consisting of κ-complete measures. This concept is due to Martin and Kechris, see [St] for details and references; however, [St] denotes Γ^κ by Hom_κ, and uses UB_κ for the pointclass of κ-Universally Baire sets.

A set of reals A is κ-weakly Homogeneous iff $A = \{\, r : \exists s\, ((r, s) \in B) \,\}$ for some κ-Homogeneous set B. Let T be a tree witnessing the κ-Homogeneity of B. Then T is called κ-weakly Homogeneous.

Definition 3.73 **(Γ^∞).** By Γ^∞ we denote the pointclass of ∞-Homogeneous sets of reals, i.e., $\Gamma^\infty = \bigcap_\kappa \Gamma^\kappa$, where the intersection runs over all cardinals.

It is a theorem of Martin and Solovay that if T is a κ-weakly Homogeneous tree, then there is a tree T^* such that T and T^* are κ-absolutely complementing. It follows from Definition 3.77 that κ-weakly Homogeneous sets of reals are κ-Universally Baire.

It is a theorem of Woodin that if δ is a Woodin cardinal and T, T^* are δ^+-absolutely complementing trees, then T is $< \delta$-weakly Homogeneous, i.e., T is α-weakly Homogeneous for all $\alpha < \delta$.

An immediate corollary of these results is that (under the assumption of a proper class of Woodin cardinals) the pointclass Γ^∞ consists precisely of the Universally Baire sets.

3.6 Real-valued measurability and the Ω-conjecture

This Section announces an improvement due to Woodin of the result in Section 3.5. We include enough definitions to make the statement meaningful.

Recall we have shown inherent difficulties to a straightforward attempt to obtain (without anti-large cardinal assumptions) extensions of the universe where $\mathfrak{c}$ is real-valued measurable and there are Δ^2_1-well-orderings of $\mathbb{R}$. The specific technical difficulty that must be resolved is whether it is possible to device a coding of bounded subsets of $\mathfrak{c}$ by reals. The usual way of obtaining such coding is by ensuring that some kind of forcing axiom holds. However, we have shown that real-valued measurability contradicts even very general schema toward such forcing axioms. The way this difficulty was dealt with in the previous Section was by circumventing it, by working within a "thin" ground model which could therefore be identified in a projective fashion in the relevant forcing extension.

Woodin's idea is to exploit this "thinness" within a more broader context. Specifically, instead of trying to establish directly that a Δ^2_1-well-ordering of $\mathbb{R}$ and $\mathsf{RVM}(\mathfrak{c})$ can be added by forcing, he settles for showing the Ω-consistency of this assumption. We proceed now to present a brief summary of Ω-logic, of Ω-consistency, of its connection with the problem of showing consistency via forcing, and close with Woodin's result and the question of possible generalizations.

In [W], Woodin introduces Ω-logic as a strong logic extending first-order logic (in fact, extending β-logic), and uses it to argue for a negative solution to Cantor's continuum problem. His argument would justify the adoption of $\neg\mathsf{CH}$ if a particular conjecture, showing that Ω-logic is as strong as possible for a wide class of statements (including CH), holds. We advise the interested reader to consult [W] for more details. All the results and definitions presented here, unless otherwise explicitly stated, are due to Woodin. However, it must be pointed out that since the appearance of [W] and even [W1], the basic definitions have changed somewhat, see [W3][20]. In particular, the definition of Ω-logic we state below is purely semantic, and corresponds to what [W1] calls Ω^*-logic. This move requires a slight change in the definition of *proofs* in Ω-logic, as we will explain.

Strong logic are defined in [W1]. We do not need this concept, but it is useful to mention that we are only interested in it with respect to theories (in a first order language) extending ZFC. Ω-logic and first order logic are both examples of strong logics, at opposite ends of the spectrum, first order logic being the most generous strong logic there is, in the sense that it allows as many structures as possible, and we regard this generosity as a

[20]This paper is partly based on professor Woodin's talk at the *International conference in Logic and Philosophy* **One Hundred Years of Rusell's Paradox**, University of Munich, Germany, June 2–5, 2001. Professor Woodin's talk was delivered on Tuesday, June 5, 4:30–5:30 pm.

weakness. On the other hand, Ω-logic is the strongest possible logic, allowing only those structures which pass for acceptable models of set theory, under reasonable requirements of acceptability. For example, when first order logic allows for any structure of the form (M, E) as a possible model, ω-logic only allows those structures which "compute V_ω correctly" and β-logic those structures which are correct about well-foundedness. Ω-logic goes as far in this direction as possible, subject to natural requirements that we list below.

Recall that if M is a transitive structure, $M_\alpha = \{ x \in M : \mathrm{rk}(x) < \alpha \}$. Suppose M satisfies enough set theory, and $\mathbb{P} \in M$ is a forcing notion. Then by $M_\alpha^{\mathbb{P}}$ we mean a name for the structure such that for any G $\mathbb{P}$-generic over M, $(M_\alpha^{\mathbb{P}})_G = (M[G])_\alpha$. As usual, we abuse language and write, for example, that $M_\alpha^{\mathbb{P}} \models \phi$, for ϕ a sentence, iff $1 \Vdash_{\mathbb{P}} M_\alpha^{\mathbb{P}} \models \phi$.

Definition 3.74 (Ω-logic). Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then

$$T \models_\Omega \phi$$

iff for all $\mathbb{P}$ and all λ, if $V_\lambda^{\mathbb{P}} \models T$, then $V_\lambda^{\mathbb{P}} \models \phi$.

Remark 3.75. According with this definition, an *Ω-satisfiable* sentence ϕ, i.e., a sentence ϕ such that $\neg\phi$ is not Ω-valid, is one such that for some $\mathbb{P}$ and α, $V_\alpha^{\mathbb{P}} \models \mathsf{ZFC} + \phi$. Taking for granted that there are enough ordinals α such that $V_\alpha \models \mathsf{ZFC}$, so this discussion is not vacuous, notice that if ϕ is Σ_2 and Ω-satisfiable, then in fact ϕ is forceable *over* V, i.e., for some $\mathbb{P}$, $V^{\mathbb{P}} \models \phi$.

A logic (in the sense of a satisfaction relation between first order structures and first order statements) satisfying the definition of Ω-logic (and, perhaps, being more restrictive) is said to be *generically sound.*

An important difference between first order logic and Ω-logic is that the latter requires a healthy large cardinal structure on the background universe for the absoluteness requirements that allow for a reasonable study of Ω-logic to hold. For this Section, let us define:

Definition 3.76. By *our Base Theory* we mean

$$\mathsf{ZFC} + \text{"There is a proper class of Woodin cardinals."}$$

Theorem 3.77 (Generic Invariance). *Assume our Base Theory. Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then $T \models_\Omega \phi$ iff for all $\mathbb{P}$, $V^{\mathbb{P}} \models T \models_\Omega \phi$.* □

Corresponding to the semantic notion of satisfiability we want to develop a syntactic counterpart, $\vdash_\Omega$. Recall that proofs in first order logic can be construed as certain trees. Similarly, for Ω-logic, we develop a notion of *certificate* that plays this role.

The certificates in this case are more specialized, and it is better to present first the sets in terms of which we are to define them, the Universally Baire sets. For our purposes, we can define the Universally Baire sets directly in the way we need them, by what is usually stated as a corollary of their standard definition:

Definition 3.78 (Feng, Magidor, Woodin [FeMaW]). A set $A \subseteq \omega^\omega$ is λ-*Universally Baire*, where λ is an infinite cardinal, iff there are λ-*absolutely complementing* trees for A, i.e., a pair T, T^* of trees on $\omega \times X$ for some X, such that

1. $A = p[T]$ and $\omega^\omega \setminus A = p[T^*]$.
2. $1 \Vdash_{\mathbb{P}} p[T] \cup p[T^*] = \omega^\omega$ for any forcing $\mathbb{P}$ of size at most λ.

A is ∞-Universally Baire or, simply, Universally Baire, iff it is λ-Universally Baire for all λ.

Notice that if A is λ-Universally Baire, and $T, T^*, \mathbb{P}$ are as above, then $1 \Vdash_{\mathbb{P}} p[T] \cap p[T^*] = \emptyset$.

The Universally Baire sets generalize the Borel sets and have all the usual regularity properties.

Under reasonable large cardinal assumptions, the pointclass of Universally Baire sets is quite closed. For example:

Fact 3.79. *Assume our Base Theory. Suppose A is Universally Baire. Then every set of reals in $L(A, \mathbb{R})$ is Universally Baire.* □

There are somewhat cleaner ways of stating this fact.

Lemma 3.80. *If every Σ^1_2-set is Universally Baire, then every set has a sharp.* □

In view of the lemma, we can present Fact 3.79 as the slightly stronger statement:

Fact 3.81. *Assume our Base Theory. Suppose A is Universally Baire. Then $A^\sharp$ is Universally Baire.* □

Given such a set A, it makes sense to talk about its *interpretation* in extensions of the universe, in what generalizes the idea of Borel codes for Borel sets.

Definition 3.82. Let A be Universally Baire. Let $\mathbb{P}$ be a forcing notion, and let G be $\mathbb{P}$-generic over V. Then the interpretation A_G of A in $V[G]$ is

$$A_G = \bigcup\{\, p[T] : T \in V \text{ and } V \models A = p[T]\,\}.$$

This is the natural notion we would expect: If T, T^* are λ-complementing trees, $|\mathbb{P}| \leq \lambda$ and G is $\mathbb{P}$-generic over V, then $V[G] \models A_G = p[T]$.

The certificates for Ω-logic are issued in terms of universally Baire sets, and thus we arrive at the concept of A-closed structures.

Definition 3.83. Let $A \subseteq \omega^\omega$ be Universally Baire. A transitive set M is *A-closed* iff for all $\mathbb{P} \in M$ and all $\mathbb{P}$-terms $\tau \in M$,

$$\{\, p \in \mathbb{P} : V \models p \Vdash \tau \in A_G \,\} \in M.$$

Remark 3.84. In practice, countable transitive A-closed models M are those admitting a pair of "absolutely complementing with respect to M" trees $T, T^* \in M$ such that the interpretation of A (which needs not be in M) would be in forcing extensions of M by forcing notions in M given by the projection of T, and such that in V, $p[T] \subseteq A$ and $p[T^*] \subseteq \mathbb{R} \setminus A$. Notice that M-generics for forcing notions in M exist in V, since M is countable.

Even though the official definition restricts the A-closed structures from the beginning to transitive sets, it may be helpful to point out that β-logic and transitive sets, are related in a similar fashion: An ω-model $(M, E) \models \mathsf{ZFC}$ is well-founded iff, under the proper interpretation, it is A-closed for each Π^1_1-set A.

The following is [W], Lemma 10.143:

Theorem 3.85 (Woodin). *Let $M \models \mathsf{ZFC}$ be transitive, and let A be Universally Baire. Then the following are equivalent:*

1. *M is A-closed.*
2. *Suppose $\mathbb{P} \in M$ and G is $\mathbb{P}$-generic over V. Then*

$$V[G] \models A_G \cap M[G] \in M[G]. \quad \square$$

With the concept of A-closed structures at hand, we are ready to define provability in Ω-logic.

Fact 3.86. *Assume our Base Theory. Let A be a Universally Baire set. Then there are A-closed countable transitive models of* ZFC. □

Definition 3.87 ($\vdash_\Omega$)**.** Let $T \supseteq \mathsf{ZFC}$ be a theory, and let ϕ be a sentence. Then

$$T \vdash_\Omega \phi$$

iff there exists a Universally Baire set A such that

1. $L(A, \mathbb{R}) \models \mathsf{AD}^+$.
2. $A^\sharp$ exists and is Universally Baire.
3. Whenever M is a countable, transitive, A-closed model of ZFC and $\alpha \in \mathrm{ORD}^M$ is such that $M_\alpha \models T$, then $M_\alpha \models \phi$.

See Section 1.4 and [W], Chapter 10 for an introduction to AD^+.

In [W], the notion we call here $\models_\Omega$ was denoted $\vdash_{\Omega^*}$ and called Ω^*-logic. Ω-logic was defined by a slight variation of 3.87, namely the infinitary implication $\bigwedge T \to \phi$ was required to hold in M itself, not in its initial segments. The change allows for a cleaner version of the Ω-conjecture, see 3.91. Originally, the Ω-conjecture needed to be stated in terms of Π_2-statements. The other difference between the definition given here and the one in [W] is due to the fact that 3.87 was stated in ZFC and not in our Base Theory. Under our Base Theory, assumptions 1. and 2. hold automatically.

One of the nicest features of $\vdash_\Omega$ is that it does not depend on the particular universe where it is considered, at least if we restrict our attention to possible generic extensions. This is the content of [W], Theorem 10.146:

Theorem 3.88 (Generic Invariance). *Assume our Base Theory. Let $T \supseteq$ ZFC and let ϕ be a sentence. Then $T \vdash_\Omega \phi$ iff for all $\mathbb{P}$, $V^{\mathbb{P}} \models T \vdash_\Omega \phi$.* □

Theorem 3.89 (Generic Soundness). *Let $T \supseteq$ ZFC and let ϕ be a sentence. Suppose $T \vdash_\Omega \phi$. Then $T \models_\Omega \phi$.* □

Remark 3.90. The previous definition of $\vdash_\Omega$ required the background assumption of our Base Theory in order for Theorem 3.89 to hold. Notice with the new definition it is stated as a ZFC result.

The Ω-conjecture is the statement that $\vdash_\Omega$ is the notion of provability associated to $\models_\Omega$ in the sense that the completeness theorem for Ω-logic holds.

Conjecture 3.91 (Ω-Conjecture). *Assume our Base Theory and let ϕ be a sentence. Then* ZFC $\models_\Omega \phi$ *iff* ZFC $\vdash_\Omega \phi$.[21]

Woodin has shown that the Ω-conjecture is true unless (in a precise sense) there are large cardinal hypothesis implying a strong failure of iterability. For example, we have the following results. For the definition of the *Weakly Homogeneous Iteration Hypothesis* WHIH we refer the reader to [W], Definition 10.4:

Lemma 3.92. *Let M be a finestructural inner model of our Base Theory. Then* WHIH *holds in M.* □

Theorem 3.93. *Assume our Base Theory. Suppose* WHIH *holds. Then the Ω-Conjecture holds in V.* □

Remark 3.94. Since the publication of [W], a cleaner description of large cardinal axioms and iterability conditions within this context has been developed in unpublished work by Hugh Woodin, see [W1].

Definition 3.95 (Ω-consistency). Assume our Base Theory. Let $T \supseteq$ ZFC and let ϕ be a sentence. Then ϕ is *Ω-consistent relative to T* (and if $T =$ ZFC, we just say ϕ is Ω-consistent) iff for any Universally Baire set A there is a countable transitive model M of $T + \phi$ which is A-closed.

Hence, at least as far as we can see nowadays, in order to prove that a proper class model of a Σ_2-sentence ϕ can be achieved (from large cardinals) by forcing, it suffices to show that for any Universally Baire set A, ϕ holds in an appropriate A-closed model M of ZFC. The intention of this comment is that it is not the same to prove that a sentence ϕ is forceable from an inner model than from the ground model itself. After all, ϕ may hold in forcing extensions of an inner model because that model is not sufficiently correct. For example, recall our results in Chapter 2. However, if the Ω-conjecture holds, and ϕ is Ω-consistent, then in fact ϕ can be forced over V.

[21] The stronger version of the conjecture asserting that $T \models_\Omega \phi$ iff $T \vdash_\Omega \phi$ holds for all theories T and sentences ϕ under the assumption of our Base Theory is expected to follow from the stated version.

Notice that any statement of the form $\exists\alpha\,(V_\alpha \models \phi)$, where ϕ is a sentence, is Σ_2, and any statement of the form $\forall\alpha\,(V_\alpha \models \phi \Rightarrow V_\alpha \models \psi)$, for ϕ and ψ sentences, is Π_2. The following follows immediately:

Fact 3.96. *The statement* "RVM($\mathfrak{c}$)+ There is a Δ^2_1-well-ordering of $\mathbb{R}$" *can be rendered in a Δ_2-way.* □

The reader should appreciate by now how powerful the Ω-conjecture is, since the witnesses to Ω-consistency of a sentence ϕ can be "finestructural-like" models, their finestructural features may be used in essential ways to establish the validity of ϕ, and nonetheless we can conclude that ϕ can be forced over the universe, without the need of any finestructural of anti-large cardinal requirements.

Since we do not know how to force a Δ^2_1-well-ordering of the reals together with RVM($\mathfrak{c}$), unless we have some nice control over the ground model itself, it was natural to attempt a proof of the Ω-consistency of this assumption. Woodin has succeeded in this attempt, and we close this Chapter with his result and a few comments.

Theorem 3.97 (Woodin). *Assume our Base Theory. Then it is Ω-consistent that $\mathfrak{c}$ is real-valued measurable and there is a Σ^2_1-well-ordering of $\mathbb{R}$.* □

Even an outline of the argument would require a serious detour through AD$^+$. The idea is to use the large cardinal assumption to produce, given a Universally Baire set A, A-closed and sufficiently "finestructure-like" inner models of strong versions[22] of AD$^+$ over which forcing with $\mathbb{P}_{\max}$ produces ZFC-models with a distinguished measurable cardinal and satisfying a strong enough fragment of PFA($\mathfrak{c}$) to ensure Lemma 1.31 holds after forcing as in Section 3.2. This provides us, combined with the finestructural features of the ground model, with an appropriate covering argument that can be used in place of Claim 3.70 to obtain the desired Σ^2_1-definition. The ground model can in fact be chosen so the forcing extension itself is A-closed, and this gives the result. The finestructural details, however, involve a new mouse hierarchy due to Woodin which remains unpublished[23].

[22]These models have the form $L_\Gamma(\mathbb{R},\mu)$, where μ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and Γ is a particular closure operator which also plays the role of the tree for Σ^2_1 inside the model.

[23]Very near to the deadline for the submission of this manuscript, Woodin has found a different proof which makes most of the outline above outdated. The new argument avoids the use of the sealing Lemma 1.31 and any appeal to the new finestructural hierarchy, and applies to a wider range of situations, for example, to the category version of real-valued measurability (κ has the property this version entails iff for some $\lambda \geq \omega_1$, $V^{\mathrm{Add}(\omega,\lambda)} \models \exists j : V \to N$, $\mathrm{cp}(j) = \kappa$) where Cohen reals and therefore Suslin trees are added

It follows immediately that granting large cardinals, if the Ω-conjecture holds, then the statement shown in 3.97 to be Ω-consistent can actually be forced. The following, however, remains open (from any large cardinal assumptions:)

Question 3.98. *Assume κ is measurable and* GCH *holds. Is there a forcing extension where $\kappa = \mathfrak{c}$ is real-valued measurable, and there is a Δ^2_1-well-ordering of $\mathbb{R}$?*

Remark 3.99. In the statement of the Ω-conjecture, our Base Theory seems necessary. Otherwise, Woodin has shown that we cannot ensure that the conjecture holds even if the pointclass of Universally Baire sets possesses strong closure properties and the universe has a sufficiently rich large cardinal structure shy from the existence of Woodin cardinals. Specifically, the following holds ([W3]):

Theorem 3.100 (Woodin). *Suppose there is a proper class of inaccessible limits of Woodin cardinals. Then there is an inner model N such that*

$$N \models \mathsf{ZFC} + \text{`` There is a proper class of inaccessible limits of Woodin cardinals ''}$$

and such that for all $\kappa < \delta$, where δ is the least Woodin of N, if κ is strongly inaccessible in N then in N_κ the following hold:

1. *$\emptyset \models_\Omega$ There are no Woodin cardinals,*
2. *$\emptyset \not\vdash_\Omega$ There are no Woodin cardinals,*
3. *For every Universally Baire set A, $A^\sharp$ exists and is Universally Baire, and $L(A, \mathbb{R}) \models$ AD^+.* □

to the model, so the sealing property actually fails. In the new proof, $\mathbb{Q}_{\max}$ is used instead of $\mathbb{P}_{\max}$ and factoring properties of the generic embeddings derived from forcing with the nonstationary ideal replace the use of Lemma 1.31.

Chapter 4

Forcing Axioms and Inner Models of GCH

Assume PFA($\mathfrak{c}$), and let M be an inner model of GCH. In this Chapter we investigate the consequences of assuming that $\aleph_2^V$ is a successor cardinal in M. This is believed to be impossible, and in particular relates to a well-known conjecture due to Cummings on collapsing successors of singulars.

4.1 Introduction

The results we present here grew out of two seemingly unrelated questions:

Question 4.1 (Woodin). *Suppose* MM *holds and M is an inner model of* GCH*. Must $\aleph_2$ be inaccessible in M?*

Question 4.2. *Are there any (consistent) assumptions under which there is a forcing extension of V that collapses $\aleph_{\omega+1}$ to $\aleph_2$?*

4.1.1 Question 4.1

Essentially, we only know one argument to show the consistency of forcing axioms like PFA or MM. For example, for PFA (Baumgartner, Shelah. See [Sh1] or [J1] for a detailed proof):

- We start with a large cardinal κ (specifically, κ is supercompact.)
- We iterate proper forcings ensuring that the final product is

  - Proper (thus, $\aleph_1$ is preserved; this is ensured by using countable support),
  - κ-cc (so κ and larger cardinals are preserved),

and

- The forcings used in the iteration are general enough to guarantee (by a reflection argument) that the final model satisfies the desired forcing axiom (this is ensured by careful bookkeeping, via a Laver function.)

When this method is carried out, we obtain a model W where

- PFA holds,
- $\aleph_1 = \aleph_1^V$, and
- $\kappa = \aleph_2$.

The method is flexible enough to allow for some variations:

- SPFA = MM is obtained by (exactly) the same argument, using semiproper forcings, and revised countable support. (Foreman, Magidor, Shelah [FoMaSh])
- Starting with a supercompact limit of supercompacts, a variant of this method allows for a model of MM where Woodin's $\mathbb{P}_{\max}$-axiom $(*)$ fails. (Larson [La])
- Starting with a strong cardinal, the method provides (via an appropriate version of Laver functions for strong cardinals) a model of SPFA($\mathfrak{c}$). (Woodin [W]. The details can be found in the proof of the result mentioned below, our Theorem 2.21.)
- Starting from finestructural models for strong cardinals, the method shows that SPFA($\mathfrak{c}$) is consistent with the existence of Σ^1_5-well-orderings of the reals.
- Bounded fragments of MM have been deduced from reflecting cardinals. (Todorčević [To2])

However, all these, and similar results, are obtained by what in essence is still the original argument for PFA.

Is this an indication of our limited understanding of forcing axioms? Or rather, does our ignorance reflect a deeper underlying reason? This apparent lack of understanding improves a little, but not significantly, when we restrict our attention to bounded fragments, like $\mathsf{MM}(\mathfrak{c})$: Except for the usual iteration, Woodin's $\mathbb{P}_{\max}$ techniques have obtained this forcing axiom by forcing over AD^+ models.

What we try to suggest here is that there might be a rich structure underneath, and the results we present might be the first steps toward its discovery.

Forcing axioms are in general quite fragile. For example, MM is destroyed by addition of a single Cohen real, and we cannot recover it without collapsing ω_2.

Fact 4.3 (Veličković [Ve]). *If* MM *holds and N is an inner model such that $\omega_2^N = \omega_2$, then $\mathcal{P}(\omega_1) \subseteq N$.* □

Corollary 4.4. *Suppose $M \models$ MM and $r \in \mathcal{P}(\omega_1) \setminus M$. Let N be an outer model of $M[r]$ such that $N \models$ MM. Then $\omega_2^N > \omega_3^M$.*

Proof: That $\omega_2^M < \omega_2^N$ is immediate from Veličković's result.

If $\omega_2^N = \omega_3^M$, then $\mathrm{cf}^N(\omega_2^M) = \omega_1^N$ by Theorem 4.9 below. But MM implies $2^{\omega_1} = \omega_2$, and therefore a cofinal set of levels of $(2^{<\omega_2})^M$ is a weak Kurepa tree in V, contradicting Baumgartner's result [Ba]. □

Question 4.5. *Can we recover* PFA *after adding a Cohen real to V without collapsing ω_2?*

Is there an essentially different technique for producing extensions of V satisfying MM? We cannot begin with a model M of GCH unless ${\aleph_2}^M < \aleph_2$, by Veličković's result. In this Chapter, we begin to explore how little leeway we actually have, at least if our original model M satisfies GCH.

We show that if MM or even weak variants of this forcing axiom hold then V cannot be a weakly proper forcing extension of an inner model M of GCH where ω_2^V is a successor cardinal. In fact, we show that if such a model M exists at all, then it exhibits a highly unlikely combinatorial structure. We expect that the scheme described above is in essence the only way of obtaining models of strong forcing axioms, which would provide a satisfactory explanation for the difficulty of obtaining a model like M. As mentioned above, Woodin's $\mathbb{P}_{\max}$ forcing and its variants show that the state of affairs is more complicated for the bounded versions of these axioms (See [W].) However, we expect that the same results carry over to this situation: Under AD^+, models like $L(\mathbb{R})^{\mathbb{P}_{\max}}$ are forcing extensions of ZFC-models like $\mathsf{HOD}^{L(\mathbb{R})}$. If V is one of these models, then ω_2^V is indeed a large cardinal in $\mathsf{HOD}^{L(\mathbb{R})}$.

4.1.2 Question 4.2

Question 4.2 is a particular instance of the more general problem of under what circumstances it is possible to collapse a singular cardinal while preserving its successor.

It is expected that the answer to 4.2 is no.

Conjecture 4.6 (Cummings [Cu]). *Assume $N \subseteq W$ are inner models, μ is a cardinal of N, and $(\mu^+)^N$ is a cardinal of W. Then $W \models \mathrm{cf}(|\mu|) = \mathrm{cf}(\mu)$.*

Observe that by König's lemma, if the question has a positive answer, in the resulting model CH must fail, since $\aleph_1^{\aleph_0} \geq \aleph_2$.

We will show that if PFA($\mathfrak{c}$) holds and there is a model M as discussed above, then Question 4.2 turns out to have a positive answer after all. Besides this situation, I know of only one argument which could perhaps provide such an outcome. I think it is due to Cummings [Cu], and it involves the following instance of Chang's conjecture, which is still open:

Question 4.7. *Is it consistent that $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$?*

In detail, this partition relation asserts that whenever $\mathcal{M} = (M, R, \ldots)$ is a structure in a countable language such that $|M| = \aleph_{\omega+1}$ and $|R| = \aleph_\omega$, there is $\mathcal{N} = (N, R \cap N, \ldots) \prec \mathcal{M}$ such that $|N| = \aleph_2$ and $|N \cap R| = \aleph_1$.

As observed by Cummings, if this instance of Chang's conjecture is consistent (with the existence of a Woodin cardinal), then

$$S = \{ X \subseteq \aleph_{\omega+1} : \mathrm{ot}(X) = \aleph_2 \}$$

is stationary, and forcing with the Stationary Tower below S gives a model where $\aleph_{\omega+1}^V$ has been collapsed to $\aleph_2$.

Notice that by an easy argument, again using König's lemma, if $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ then CH fails.

4.2 ω_1^V in inner models of GCH

Throughout this Chapter, unless explicitly stated, all the inner models we consider are models of choice.

From now on, assume PFA($\mathfrak{c}$) holds and fix an inner model M of GCH. We show here that in addition we can assume that M computes $\aleph_1$ correctly.

Claim 4.8. *Assume* PFA($\mathfrak{c}$), *and let* $M \models$ GCH *be an inner model. Then, without loss of generality,* $\omega_1^M = \omega_1^V$. *More carefully, there is an inner model* N *with* $M \subseteq N \subseteq V$ *such that* $N \models$ GCH, $\omega_1^N = \omega_1^V$, *and* ω_2^V *is inaccessible in* N *iff it is inaccessible in* M.

Proof: Let $\kappa = \omega_1^V$, and $\mathbb{P} = \mathtt{Coll}(\omega, < \kappa)$. Then $\mathbb{P} \in M$ and $\mathbb{P}$ is ccc.

The following is contained in [Sh], Lemma VII.4.9:

Theorem 4.9 (Shelah). *Suppose* $N \subseteq W$ *are inner models and* μ *is a regular cardinal of* N.

Suppose $(\mu^+)^N$ *is a cardinal of* W. *Then* $W \models \mathrm{cf}(|\mu|) = \mathrm{cf}(\mu)$. $\triangle$

Shelah's Theorem 4.9 is the key to prove the following:

Lemma 4.10. *Suppose* ω_2^V *is a successor in* M, *say* $\omega_2^V = (\lambda^+)^M$. *Then* λ *is singular in* M.

Proof: By Shelah's result, it suffices to prove that $\mathrm{cf}(\lambda) \neq \omega_1$ in V.

Claim 4.11. $\mathrm{cf}(\lambda) = \omega$.

Proof: Proceed by contradiction, as in the last paragraph of the proof of Corollary 4.4. $\triangledown$ $\triangle$

The argument splits now into two cases, according to whether ω_2^V is inaccessible in M or not.

If $\omega_2^V = (\lambda^+)^M$, then by the lemma λ is singular, thus limit, in M. In particular, it is bigger than $(\kappa^+)^M$. Otherwise, ω_2^V is inaccessible in M. In both cases, it is certainly bigger than $(\kappa^+)^M = |\mathcal{P}(\mathbb{P})|^M$, so $|\mathcal{P}^M(\mathbb{P})| = \aleph_1$. By $\mathsf{MA}_{\aleph_1}$ there is a $\mathbb{P}$-generic G over M. In $M[G]$, GCH holds, $\omega_1^{M[G]} = \omega_1$, and if $\omega_2^V = (\lambda^+)^M$, then λ is preserved since $\mathbb{P}$ is κ-cc in M. $\square$

In view of the claim, from now on we assume in addition that $\omega_1^V = \omega_1^M$, that λ is a cardinal of M, and that $\omega_2^V = (\lambda^+)^M$. By Claim 4.11, $\mathrm{cf}^V(\lambda) = \omega$. By Lemma 4.10, λ is singular in M.

Remark 4.12. At the cost of assuming a stronger forcing axiom, there is a proof of Lemma 4.10 that avoids Shelah's result: Assume PFA. If λ is regular in M, then $M \models \lambda^{<\lambda} = \lambda$, by GCH. By Specker's theorem [Sp], in M there is a special λ^+-tree. In particular, this is an $\aleph_2$-Aronszajn tree in V. This contradicts PFA, by Todorčević's result [To] (See also [To2].)

From this proof we have:

Corollary 4.13. *If* PFA *holds, then* $M \models$ `There are no` λ^+`-special trees.` □

We will revisit Theorem 4.9 in the next Section.

4.3 Almost Disjoint Sequences

The goal of this Section is to show that $\mathrm{cf}(\lambda) = \omega$ also holds in M.

In fact we do not require a forcing axiom in V and instead only assume that $M \models$ GCH, $\omega_1^V = \omega_1^M$, and $\omega_2^V = (\lambda^+)^M$, for λ a singular cardinal in M such that $\mathrm{cf}^V(\lambda) = \omega$, so $\mathrm{cf}(\lambda) < \mathrm{cf}(|\lambda|)$. Notice that this is precisely the kind of situation Cummings's conjecture 4.6 indicates is impossible.

Lemma 4.14. $M \models \mathrm{cf}(\lambda) = \omega$.

For this we build on results of Džamonja and Shelah (see [Sh] and [DžSh]).

Definition 4.15. Let κ be a cardinal.

1. A *uniformly almost disjoint sequence* for κ is a sequence $\langle A_\alpha : \alpha < \kappa^+ \rangle$ of unbounded subsets of κ for which there are functions $\langle f_\beta : \beta < \kappa^+ \rangle$ such that for all $\beta < \kappa^+$
   - $f_\beta : \beta \to \kappa$, and
   - $\langle A_\alpha \setminus f_\beta(\alpha) : \alpha < \beta \rangle$ is a sequence of pairwise disjoint sets.
2. ADS_κ holds iff there is a uniformly almost disjoint sequence for κ.

Lemma 4.16 (Shelah [Sh]). *If $N \subseteq W$ are inner models, μ is a cardinal of N, $W \models \mathrm{cf}(|\mu|) \neq \mathrm{cf}(\mu)$, and ADS_μ holds in N, then $(\mu^+)^N$ is not a cardinal of W.* □

Corollary 4.17. $M \models \mathsf{ADS}_\lambda$ `fails.` □

Shelah ([Sh], Lemma VII.4.9.) also shows that ADS_μ holds if μ is regular or if $\square_\mu$ holds. Namely, suppose μ is singular, and fix an increasing sequence of regular cardinals cofinal in μ, $\langle \mu_i : i \in \mathrm{cf}(\mu) \rangle$. Let $\langle C_\alpha : \alpha < \mu^+ \rangle$ be a $\square_\mu$-sequence, i.e., for all $\alpha < \mu^+$,

- $C_\alpha \subseteq \alpha$ is club,
- $\forall \beta \in \mathrm{acc}(C_\alpha)\, (C_\beta = \beta \cap C_\alpha)$, and

- $\mathrm{ot}(C_\alpha) \leq \mu$.

Inductively define $\langle g_\alpha : \alpha < \mu^+ \rangle$ so $g_\alpha \in \prod_i \mu_i$ for each $\alpha < \mu^+$, the sequence is $<^*$-increasing, where $<^*$ is the eventual domination relation:

$$f <^* g \iff \exists i \, \forall j \geq i \, \big(f(j) < g(j)\big),$$

and if $\alpha \in C_\beta$ and $|C_\beta| < \mu_i$, then $g_\alpha(i) < g_\beta(i)$.

Set $A_\alpha = \mathrm{ran}(g_\alpha)$. Then $\langle A_\alpha : \alpha < \mu^+ \rangle$ is a uniformly almost disjoint sequence for μ.

Cummings, Foreman and Magidor [CuFoMa] improve Shelah's result by showing that $\Box^*_\mu$ suffices. Here we notice as an immediate corollary of results of [DžSh] that ZFC suffices to show ADS_μ holds for many singular cardinals μ.

Definition 4.18. Let τ be an ordinal and S be a set of ordinals. Then S *admits squares type-bounded by* τ iff there is a sequence $\langle C_\alpha : \alpha \in S \rangle$ such that for all $\alpha \in S$:

- $C_\alpha \subseteq \alpha \cap S$ is closed in α. It is unbounded in α if α is limit.
- $\mathrm{ot}(C_\alpha) < \tau$.
- For all $\beta \in \mathrm{acc}(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.

Theorem 4.19 (Džamonja and Shelah [DžSh]). *Suppose that μ is singular, $\nu = \mu^+$, and $\omega < \mathrm{cf}(\kappa) = \kappa < \mu$ is such that $\mathrm{cf}\big(\mathcal{P}_{\kappa^+}(\mu), \subseteq\big) = \mu$. Then*

$$\{\, \alpha < \nu : \mathrm{cf}(\alpha) \leq \kappa \,\}$$

is the union of μ sets each of which admits squares type-bounded by κ^+. □

Corollary 4.20. *Suppose μ is singular, $\omega < \mathrm{cf}(\kappa) = \kappa < \mu$, and $\mathrm{cf}\big(\mathcal{P}_{\kappa^+}(\mu), \subseteq\big) = \mu$. Then ADS_μ holds.*

Proof: A trivial modification of Shelah's argument works. Rather than a sequence of functions indexed by all ordinals below μ^+, build a sequence indexed by a set admitting squares. □

Proof of Lemma 4.14: Recall that λ is singular in M. If the lemma fails then, since $\mathrm{cf}(\lambda) = \omega$ in V, we must have $M \models \mathrm{cf}(\lambda) > \omega_1$ (recall that $\omega_1^V = \omega_1^M$). By GCH,

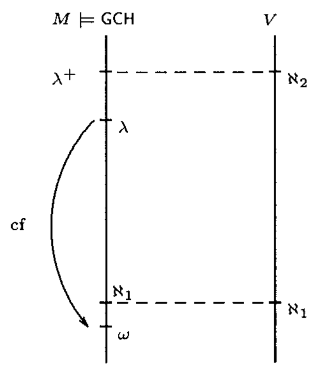

Figure 4.1: $M \subset V$, $M \models \mathsf{GCH}$, $\aleph_1^V = \aleph_1^M$, $\aleph_2^V = (\lambda^+)^M$, $\mathrm{cf}^V(\lambda) = \omega$

$M \models \mathrm{cf}(\mathcal{P}_{\aleph_2^M}(\lambda), \subseteq) = \lambda$. Then ADS_λ holds in M, and by Shelah's result $(\lambda^+)^M$ is not a cardinal of V, contradiction. □

The argument of course applies to much more general situations than the one concerning us. In particular, no appeal to forcing axioms was made throughout this Section.

4.4 Good points

In this Section we build on results of Cummings, Foreman and Magidor (see [FoMa] and [Cu]) to add to the main lemma of Section 4.3. Our main result is[1]:

Lemma 4.21. $M \models$ `The approachability property fails at` λ. `In fact, if` $\lambda = \aleph_\omega$, `then` VWS_λ `fails.`

This is contained in essentially known results, as the reader will appreciate from the argument to follow. We could not find an easy to trace proof in the literature, so we are including a sketch here. In fact, we obtain more than this. We try to be reasonably self-contained.

Definition 4.22. Let $\kappa > \mathrm{cf}(\kappa)$. A *scale* of length κ^+ for κ (or, simply, a scale for κ) is a pair $(\vec{\kappa}, \vec{f})$ such that $\vec{\kappa} = \langle \kappa_i : i \in \mathrm{cf}(\kappa) \rangle$ is a strictly increasing sequence of regular cardinals cofinal in κ and $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is a sequence of functions in $\prod_i \kappa_i$ which is strictly increasing and cofinal with respect to the eventual domination relation, $<^*$. [If no danger of confusion arises, we abuse language and refer to the scale $\vec{f}$.]

[1]See 4.31 for the definitions of the approachability property and of VWS_λ.

A key fact in PCF theory is the following result; it is immediate from GCH:

Theorem 4.23 (Shelah [Sh]). *Let κ be singular. Then there is a scale for κ.* $\square$

Definition 4.24. Let δ, κ be ordinals, and let $\vec{f} = \langle f_\alpha : \alpha < \eta \rangle$ be a $<^*$-strictly increasing sequence of functions $f_\alpha : \delta \to \kappa$. An ordinal $\alpha < \eta$ is *good* (with respect to $\vec{f}$) iff there is an unbounded set $A \subset \alpha$ and an $i < \delta$ such that

$$\forall \beta, \gamma \in A \, \forall j \geq i \left(\beta < \gamma \Rightarrow f_\beta(j) < f_\gamma(j)\right).$$

This notion is dubbed *flatness* in [Ko], where it is only considered for $\mathrm{cf}(\alpha) > \mathrm{cf}(\delta)$. Notice that for $\vec{f}$ as in the definition, α is good whenever $\mathrm{cf}(\alpha) < \mathrm{cf}(\delta)$.

Fact 4.25 (Shelah). *Let κ be singular, let $\rho < \kappa$ be regular, and let $(\vec{\kappa}, \vec{f})$ be a scale for κ. Then $\{\, \alpha < \kappa^+ : \mathrm{cf}(\alpha) = \rho$ and α is good$\}$ is stationary in κ^+.* $\square$

Theorem 4.26 (Cummings). *Let $N \subset W$ be inner models contradicting Cummings's conjecture at μ. Suppose $(\mu^+)^N = (\nu^+)^W$ and $\nu > \omega$. Let $(\vec{\mu}, \vec{f})$ be a scale for μ in N. Then there is $\eta < \nu$ such that whenever $\eta < \delta \leq \nu$ is regular,*

$$\{\, \gamma < \nu^+ : \mathrm{cf}(\gamma) = \delta \text{ and } \gamma \text{ is good}\}$$

is nonstationary in W. $\square$

We sketch part of the argument of Cummings's theorem in our situation. Remember $M \models$ GCH is an inner model, $\omega_1^V = \omega_1^M$, $\omega_2^V = (\lambda^+)^M$, and $\mathrm{cf}^M(\lambda) = \omega$.

Fix a scale $(\vec{\lambda}, \vec{f})$ for λ in M.

Write $\lambda = \bigcup_{\gamma<\omega_1} X_\gamma$, where $\langle X_\gamma : \gamma < \omega_1 \rangle$ is an increasing sequence of countable sets.

For $\gamma < \omega_1$, set

$$B_\gamma := \{\, \alpha < (\lambda^+)^M : \mathrm{ran}(f_\alpha) \subseteq X_\gamma \,\},$$

and notice that $\langle B_\gamma : \gamma < \omega_1 \rangle$ is increasing and $\omega_2^V = \bigcup_{\gamma<\omega_1} B_\gamma$. Thus, for all γ sufficiently large (say $\gamma \geq \gamma_0$), B_γ is unbounded.

For any such γ, let $D_\gamma := \mathrm{acc}(B_\gamma)$. Cummings concludes by arguing that no $\delta \in D_\gamma \cap S^{\omega_2^V}_{\omega_1}$ is good: Otherwise, X_γ would be uncountable.

In our situation, this immediately gives

Corollary 4.27. $\{\gamma < \omega_2 : \mathrm{cf}(\gamma) = \omega_1,\ \gamma \textit{ is good}\}$ *is nonstationary.* $\square$

Remark 4.28. In [W], Definition 10.60, Woodin introduces the concept of weak properness:

Definition 4.29. A partial order $\mathbb{P}$ is *weakly proper* iff every countable set of ordinals in $V^{\mathbb{P}}$ is covered by a countable set in V.

Notice that every proper forcing is weakly proper, and that if $\mathbb{P} \cong \mathbb{Q} * \dot{\mathbb{R}}$ is weakly proper then so is $\mathbb{Q}$, and $\dot{\mathbb{R}}$ is weakly proper in $V^{\mathbb{Q}}$.

By Fact 4.25, if γ and X_γ are defined as above, then X_γ is a countable set such that no countable set in M can cover. In particular, we have the following:

Corollary 4.30. *V is not a forcing extension of M via weakly proper forcing.* $\square$

Similarly, suppose for example that W is an inner model and κ is supercompact in W. Let $\mathbb{P}$ be in W the standard forcing of size κ to make PFA hold, let G be $\mathbb{P}$-generic over W, and set $V_1 = W[G]$ (so $\kappa = \aleph_2^{V_1}$). Then no M as required can be intermediate between W and V_1. Otherwise, V_1 would be a forcing extension of M and, since $\mathbb{P}$ is proper, the extension from M to V_1 would be weakly proper, and we reach a contradiction.

And the variations, as indicated in Section 4.1 are endless. For example, the same holds if κ is strong in W and $\mathbb{P}$ is the standard forcing of size κ to make SPFA($\mathfrak{c}$) hold.

Definition 4.31 (Foreman, Magidor [FoMa]). Let κ be a cardinal.

1. An *approachability sequence* for κ is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that for a club of $\alpha \in \kappa^+$,
    - $C_\alpha \subseteq \alpha$ is unbounded,
    - $\mathrm{ot}(C_\alpha) = \mathrm{cf}(\alpha)$, and
    - $\forall \beta < \alpha\, \exists \gamma < \alpha\, (C_\alpha \cap \beta = C_\gamma)$.
2. *The approachability property* holds at κ iff there is an approachability sequence for κ.
3. A *very weak square sequence* for κ is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that for a club of $\alpha \in \kappa^+$,
    - $C_\alpha \subseteq \alpha$ is unbounded, and

- $\forall x \in \mathcal{P}_{\omega_1}(C_\alpha)\,\big(x$ bounded in $C_\alpha \Rightarrow \exists\beta < \alpha\,(x = C_\beta)\big)$.

4. *Very weak square* holds at κ, VWS_κ, iff there is a very weak square sequence for κ.

That Lemma 4.21 relates to the result from Section 4.3 follows from *(1)* of the following lemma:

Lemma 4.32 (Foreman, Magidor [FoMa]). *Let κ be a strong limit singular cardinal.*

1. *If* $\mathrm{cf}(\kappa) > \omega$*, then* VWS_κ *holds.*
2. *Let $(\vec{\kappa}, \vec{f})$ be a scale for κ. Suppose VWS_κ holds. Then there is a club $\mathcal{C}$ such that all $\gamma \in \mathcal{C} \cap S^{\kappa^+}_{\omega_1}$ are good.*
3. *Suppose the approachability property holds. Then there is a club $\mathcal{C}$ such that all $\gamma \in \mathcal{C} \setminus S^{\kappa^+}_{\omega}$ are good.* $\square$

Hence, the approachability property fails in M at λ, and either very weak square fails in M at λ, or $(S^{\lambda^+}_{\omega_1})^M$ is disjoint from $D_\gamma \cap \mathcal{C}$, where $\mathcal{C} \in M$ is as in the lemma with $\kappa = \lambda$ and γ is such that B_γ is unbounded.

Lemma 4.21 follows now from results in [Ko]. First, we need a definition.

Definition 4.33. Let I be an ideal over a set A. A function $g : A \to \mathrm{ORD}$ is an exact upper bound (an *eub*) of $F \subseteq {}^A\mathrm{ORD}$ (with respect to I) iff

1. $\forall f \in F\,(f \leq_I g)$, and
2. For any $g' : A \to \mathrm{ORD}$, if $g' <_I g$ then $\exists f \in F\,(g' <_I f)$.

Theorem 4.34 (Shelah's trichotomy theorem, see [Ko]). *Suppose $\rho > |A|^+$ is regular, I is an ideal over A and $\vec{f} = \langle f_\alpha : \alpha < \rho \rangle$ is an $<_I$-increasing sequence of ordinal functions on A. Then $\vec{f}$ satisfies one of the following:*

- *(Good) $\vec{f}$ has an eub f with $\mathrm{cf}\, f(a) > |A|$ for all $a \in A$;*
- *(Bad) There are sets $S(a)$ for $a \in A$ satisfying $|S(a)| \leq |A|$ and an ultrafilter $\mathcal{U}$ over A extending the dual of I so that for all $\alpha < \rho$ there exists $h_\alpha \in \prod_a S(a)$ and $\beta < \rho$ such that $f_\alpha <_{\mathcal{U}} h_\alpha <_{\mathcal{U}} f_\beta$.*
- *(Ugly) There is a function $g : A \to \mathrm{ORD}$ such that the sequence $\vec{t} = \langle t_\alpha : \alpha < \rho \rangle$ does not stabilize modulo I, where $t_\alpha = \{\, a \in A : f_\alpha(a) > g(a) \,\}$.* $\square$

Remark 4.35. Notice that if *Ugly* or *Bad* applies in the situation described in the trichotomy theorem, then $2^{|A|} \geq \rho$, so $2^{|A|} > |A|^+$.

Lemma 4.36 (Kojman [Ko]). *Suppose that* $\rho > |A|^+$, I *is an ideal over* A, $\vec{f} = \langle f_\alpha : \alpha < \rho \rangle \subseteq {}^A\mathrm{ORD}$ *is* $<_I$*-increasing,* f *is an eub of* $\vec{f}$ *and* $\liminf_I \mathrm{cf}\, f(a) = \mu$. *If* $\kappa = \mathrm{cf}\,\kappa < \mu$ *and* $\theta^{|A|} < \kappa$ *for all* $\theta < \kappa$, *then every point of cofinality* κ *in* $\vec{f}$ *is good.* □

Corollary 4.37. $M \models \forall \alpha < \lambda^+ \big($`If` $\mathrm{cf}(\alpha)$ `is not the successor of a cardinal of cofinality` ω `then` α `is good`$\big)$.

Proof: Let I be the Frechet ideal over ω, that is, the ideal of finite sets. Notice that by CH and the trichotomy theorem, the scale $\vec{f}$ has an eub, and since each $f_\alpha \in \prod_n \lambda_n$, it is easy to see that $\lambda = \liminf_I \mathrm{cf}\, f(a)$. The result follows immediately from the previous lemma. □

Proof of Lemma 4.21: From 4.37 and Cummings's result it follows that if $\lambda = \aleph_\omega^M$, then $(S^{\omega_2^V}_{\omega_1})^M$ is stationary and, in fact, M correctly computes the cofinality of almost every $\alpha \in S^{\omega_2}_{\omega_1}$.

The remark following Lemma 4.32 completes the proof. □

What can be said about the points of cofinality ω? For example, is it necessarily the case that $\{\, \alpha \in S^{\omega_2}_{\omega} : \alpha \text{ is good} \,\}$ is stationary?

This question has the flavor of a problem in partition calculus. At the same time, some new idea seems to be required. For example, notice the following:

Remark 4.38. *Suppose that* $\vec{g} = \langle g_\alpha : \alpha < \mu \rangle$ *is a* $<^*$*-increasing sequence of functions* $g_\alpha : \omega \to \mathrm{ORD}$, *and that* $\mathfrak{c} < \mathrm{cf}\,\mu$. *Then both the set of good points with respect to* $\vec{g}$ *of cofinality* ω *and the set of good points of cofinality* ω_1 *are stationary in* μ.

Proof: Let $C \subseteq \mu$ be a club, and define $h : [C]^2 \to \omega$ by $h(\alpha, \beta)_< = n$ iff n is least such that $\forall m \geq n \,\big(f_\alpha(m) < f_\beta(m)\big)$.

By the Erdős-Rado theorem there is an uncountable homogeneous subset of C. Let γ be the supremum of its first ω many members. Then $\gamma \in C$ is good and has cofinality ω. Let δ be the supremum of its first ω_1 many members. Then $\delta \in C$ is good and has cofinality ω_1. □

However, this remark does not apply in our case: By König's lemma, there can be no inner model of V extending M which still satisfies CH and computes ω_2 correctly (the set of good points of cofinality ω would be stationary in this model.)

An easy observation, with which we conclude this Section, is that if $\mathsf{PFA}(\mathfrak{c})$ holds and M is an inner model of GCH that computes $\aleph_1$ correctly and where $\aleph_2^V$ is a successor, then there is a real r such that $M[r]$ computes $\aleph_2$ correctly. It follows that CH must fail in $M[r]$, since otherwise $(2^{<\omega_1})^{M[r]}$ would be a weak Kurepa tree in V.

In effect, suppose $\aleph_2^V = (\lambda^+)^M$. Let t be a subset of ω_1 coding a well-ordering in order-type λ. Then $M[t]$ computes $\aleph_2$ correctly. Let $\vec{r} = (r_\alpha : \alpha < \omega_1)$ be a sequence of almost-disjoint reals in M. By MA_{ω_1}, there is a real r coding t (in the sense of almost-disjoint forcing, see Chapter 2) with respect to $\vec{r}$. Then $t \in M[r]$, and we are done.

Question 4.39 (Woodin). *Suppose $\underset{\sim}{\delta}^1_2 = \aleph_2$ and $\forall r \in \mathbb{R}\,(r^\sharp$ exists) (this is implied, for example, by $\mathsf{MM}(\mathfrak{c})$). Let M be an inner model correctly computing $\aleph_2$. Must CH fail in M?*

Notice that the hypothesis of 4.39 implies the existence of reals r such that $M[r] \models \neg\mathsf{CH}$. However, the arguments given above do not apply to this case, since now V may have weak Kurepa trees.

Question 4.40 (Woodin). *Suppose V is closed under sharps. Let M be an inner model correctly computing $\aleph_2$. Does $(\underset{\sim}{\delta}^1_2)^M = \underset{\sim}{\delta}^1_2$?*

4.5 A cautionary remark

An overly enthusiastic previous incarnation of this Chapter actually claimed to have solved Woodin's question, and to show that if Question 4.2 has a positive answer, then GCH fails in the ground model.

These problems are still open. However, a published result of Shelah, [Sh] Claim VII.4.19, led us to their solution. Unfortunately, not all the hypotheses of this Claim are stated in [Sh]. This is mentioned here so the reader avoids following us into the same pitfall. The statement below is the result actually proved in [Sh].

Claim 4.41 (Shelah). *Suppose $N \subsetneq W$ are inner models, $W = N[r]$, $r \subseteq \mu$, μ a cardinal of N, $(\mu^+)^N = \aleph_2^W$, and $N \models$ GCH. Then $W \models$ CH,* provided that $0^\sharp$ does not exist. □

In [Sh], Claim 4.41 is stated without the $0^\sharp$ assumption. It is relatively straightforward to solve Woodin's question and Question 4.2 under GCH with the statement as it appears in [Sh].

However, as shown in the previous Sections, $0^\sharp$ exists (and much more) if either question has a negative answer.

We thank Ralf Schindler for pointing out to us that 4.41 must fail without some anti-large cardinal assumption[2]. An examination of the proof in [Sh] shows that some such assumption ($0^\sharp$ being a natural candidate, given the result to which the Claim is applied) is tacitly assumed. For example:

- First, it is claimed that $\mathrm{cf}^W(\mu) = \omega_1$.
- More seriously still, an appeal is made to what Shelah calls the strong $\aleph_2$-covering lemma, which it is claimed must hold between an appropriate inner model of N and W. This is a very strong assumption. Basically, the result requires that the sets of the smaller model are stationary in the bigger one.

Notice that both of these claims fail in the case that concerns us, the first one by Claim 4.11, and the second by Cummings's argument leading to Corollary 4.30.

As a matter of fact, a result of Woodin and Shelah contradicts the Claim, as stated in [Sh]:

Theorem 4.42 (Shelah, Woodin [ShW]). *Assume* ZFC + GCH. *Suppose that there are μ many measurable cardinals, $\mu > \aleph_1$ a regular cardinal. Then there is a pair (N, W) of cardinal-preserving class generic extensions of the universe such that*

- $W = N[r]$, *r a real,*
- $N \models$ GCH, *and*
- $W \models \mathfrak{c} = \mu$. □

This is accomplished via, among others, a Prikry-style partial order, thus producing an extension where covering must fail.

[2]From the results in [Sh] it follows that the assumption can be relaxed to "There are no inner models with measurable cardinals."

"Whatever is almost true is quite false, and among the most dangerous of errors, because being so near truth, it is more likely to lead astray."

Henry Ward Beecher (1813–1887)

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Professor Woodin's talk was the first of a cycle of three Coxeter Lectures, delivered on November 4–6, 2002, 3:30–5:00 pm. Slides available (as of this writting) at

`http://av.fields.utoronto.ca/slides/02-03/coxeter_lectures/woodin/download.pdf`

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