DETERMINACY AND JÓNSSON CARDINALS

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Abstract. We use the HOD analysis to show that every cardinal below Θ is Jónsson under the assumption of AD^+.

Contents

1. Introduction

It is a celebrated result of Kleinberg that AD, the axiom of determinacy, implies that the $\aleph_n$ are Jónsson cardinals and $\aleph_\omega$ is in fact Rowbottom; this follows from a careful analysis of partition properties and the computation of ultrapowers of associated measures, see for example [Kle77]. Kleinberg's techniques can be extended beyond $\aleph_\omega$, see for example [Lö02]. In fact, using his theory of descriptions of measures, Steve Jackson proved (in unpublished work) that, assuming determinacy, every cardinal below $\aleph_{\omega_1}$ is Rowbottom. Woodin mentioned after attending a talk on this result that the HOD analysis shows that every cardinal is Jónsson below Θ.

During the Second Conference on the Core Model Induction and HOD Mice at Münster, at my insistence, Jackson, Ketchersid, and Schlutzenberg reconstructed what we believe is Woodin's argument. Here is an account of the proof. This note is hastily written. Please email me any additions/comments/corrections/suggestions you find appropriate.

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2. Jónsson cardinals

Theorem 1 (AD). *Every cardinal κ in $L(\mathbb{R})$ below Θ satisfies:*

(1) $\mathrm{cf}(\kappa)=\omega$ *and κ is Rowbottom:*

$$\kappa \to [\kappa]_{\lambda,<\omega_1} \quad \forall\lambda<\kappa$$

or

(2) $\mathrm{cf}(\kappa)>\omega$ *and*

$$\kappa \to [\kappa]_{\lambda,<\omega_1} \quad \forall\lambda<\mathrm{cf}(\kappa)$$

(so, in particular, κ is Jónsson).

The theorem follows from the following:

Claim 2 (ZFC). *Suppose that κ is a limit of measurable cardinals. Then one of the following is the case:*

(1) $\mathrm{cf}(\kappa)=\omega$ *and κ is Rowbottom, as in the conclusion of (1) above.*

(2) $\mathrm{cf}(\kappa)$ *is measurable, κ is singular, and the conclusion of (2) above holds.*

(3) κ is measurable.

Fact 3 (AD). *Every cardinal in $L(\mathbb{R})$ is either measurable or a limit of measurables in HOD_x for any real x.*

The theorem follows easily:

Proof. Fix $f:[\kappa]^{<\omega}\to\lambda$ in HOD_x, using that any f is OD_x for some real x. Choose x so HOD_x also contains a sequence witnessing the true cofinality of κ. Use the fact to obtain an appropriately homogeneous set $H\in[\kappa]^\kappa$ in HOD_x witnessing (1) or (2). But then H works in V as well. □

The claim can be established by a direct inductive argument:

Proof. We concentrate on case 2. Let $f:[\kappa]^{<\omega}\to\lambda<\mathrm{cf}(\kappa)=\rho$ be given. We assume that ρ is measurable and fix a sequence $(\kappa_\xi:\xi<\rho)$ of measurables converging to κ, $\rho<\kappa_0$. Let $B_\xi=[\kappa_\xi,\kappa_{\xi+1})$. We produce the required homogeneous set by an inductive process that thins out the blocks B_ξ. Start by fixing measures on the κ_ξ and ρ.

Step 1. Use Rowbottom's theorem (see [Kan03, Theorem 7.17]) to find $A^0_\xi \subseteq B_\xi$ that have measure 1 in the appropriate measures and such that

$$|f[A^0_\xi]^n| = 1$$

for all n.

Step $n+1$. By an inductive procedure of length ρ, thin out each $A^n_{\xi_1}$ to produce measure 1 sets $A^{n+1}_{\xi_1}$ such that for any $m_1 < \cdots < m_{n+1}$ and any $\xi_2 < \cdots < \xi_{n+1}$ with $\xi_1 < \xi_2$ we have

$$|f[[A^{n+1}_{\xi_1}]^{m_1} \otimes \cdots \otimes [A^{n+1}_{\xi_{n+1}}]^{m_{n+1}}]| = 1.$$

Step ω. Let $A^\omega_\xi = \bigcap_n A^n_\xi$ for each $\xi < \rho$. We have that for each n, all $m_1 < \cdots < m_n$, and all $\xi_1 < \cdots < \xi_n$, we have

$$|f[[A^\omega_{\xi_1}]^{m_1} \otimes \cdots \otimes [A^\omega_{\xi_n}]^{m_n}]| = 1.$$

Now use the measurability of ρ to find a measure 1 set A of indices ξ such that for any n and $m_1 < \cdots < m_n$, there is a fixed color $c_{m_1,\dots,m_n}$ such that whenever $\xi_1 < \cdots < \xi_n$ are in A, we have

$$f[[A^\omega_{\xi_1}]^{m_1} \otimes \cdots \otimes [A^\omega_{\xi_n}]^{m_n}] = c_{m_1,\dots,m_n}.$$

It follows that $H = \bigcup_{\xi \in A} A^\omega_\xi$ is the homogeneous set we were looking for.

Case 1 is even easier, as the initial thinning gives already ω many colors, and we do not need the last step. □

We are left with the proof of the fact, to which we now proceed:

Proof. We use the directed system analysis of HOD. For this we require a fine structural fact:

Lemma 4. *Assume $\mathcal{T}$ is a stack of normal trees on a (fine structural) premouse $\mathcal{M}$, b is a nondropping cofinal branch, H is a proper class model of* ZFC*, $(\mathcal{M}, \mathcal{T}, b) \in H$, $\kappa \in \mathcal{M}^\mathcal{T}_b$, $i^\mathcal{T}_b : \mathcal{M} \to \mathcal{M}^\mathcal{T}_b$ is the associated embedding, κ is in the range of $i^\mathcal{T}_b$, $i^\mathcal{T}_b(\bar\kappa) = \kappa$, and $\mathcal{M}$ is countable in H.*

If $H \models (|\mathcal{M}| < \kappa$ and κ is a cardinal$)$, then $\bar\kappa$ is measurable or limit of measurables in $\mathcal{M}$.

Proof. We sketch the argument. By contradiction, let

$$\bar\lambda = \sup\{\gamma \le \bar\kappa : \gamma \text{ is measurable in } \mathcal{M}\},$$

and assume that $\bar\lambda < \bar\kappa$.

We consider the simple case where $\mathcal{T}$ is a single tree, and all ultrapowers are Σ_0 ultrapowers, ignoring additional fine structural issues. Inductively, we argue that if $\alpha \in b$, then

$$(\bar{\kappa_\alpha}^+)^{\mathcal{M}_\alpha} \subseteq \mathrm{Hull}_0^{\mathcal{M}_\alpha}(i^{\mathcal{T}}_{0,\alpha}[\mathcal{M}] \cup \bar{\lambda_\alpha}), \tag{1}$$

where $\bar\kappa_\alpha$ is the image of $\bar\kappa$ in $\mathcal{M}_\alpha$ under the iteration map and similarly for $\bar\lambda_\alpha$.

This is trivial at the base and at limit stages, so only the inductive step needs arguing. Let E_α be the extender used at stage α. Let α^* be the index of the model whose ultrapower by E_α gives us $\mathcal{M}_{\alpha+1}$. We may assume E_α overlaps $\bar{\kappa_\alpha}$ or there is nothing to prove. Then $\gamma_\alpha := \mathrm{cr}(E_\alpha) < \bar\lambda_\alpha$. We have

$$(\bar{\kappa_{\alpha^*}}^+)^{\mathcal{M}_{\alpha^*}} \subseteq \mathrm{Hull}_0^{\mathcal{M}_{\alpha^*}}(i^{\mathcal{T}}_{0,\alpha^*}[\mathcal{M}] \cup \bar{\lambda_{\alpha^*}}),$$

and we want to show that any $\xi < (\bar\kappa^+_{\alpha+1})^{\mathcal{M}^{\mathcal{T}}_{\alpha+1}}$ will get in the corresponding hull.

Any such ξ is represented in the ultrapower by some

$$f : [\gamma_\alpha]^{|a|} \to (\bar{\kappa_{\alpha^*}}^+)^{\mathcal{M}_{\alpha^*}}$$

where $a \in [\mathrm{lh}(E_\alpha)]^{<\omega}$, $[a,f]^{\mathcal{M}_{\alpha^*}}_{E_\alpha} = \xi$. Note that $\sup(a) < \bar{\lambda_{\alpha+1}}$. Also, f is definable from $i^{\mathcal{T}}_{0\alpha^*}(p) \cup \bar\eta$ for some $p \in \mathcal{M}$ and some $\bar\eta \in \lambda^{<\omega}_{\alpha^*}$, by induction, because whenever $(\kappa^+)^{\mathcal{M}} \subseteq \mathrm{Hull}_0^{\mathcal{M}}(X)$ for some $\mathcal{M}$ and some X, then

$$\mathcal{M}\|(\kappa^+)^{\mathcal{M}} \subseteq \mathrm{Hull}_0^{\mathcal{M}}(X) \tag{2}$$

as well, using that $\mathcal{M}$ is fine structural.

It follows that ξ can be recovered as $i_{\alpha^*,\alpha+1}(f)(a)$, since this is definable from a and $i^{\mathcal{T}}_{0,\alpha+1}(p) \cup i_{\alpha^*,\alpha+1}(\bar\eta)$.

This completes the proof of (1).

Remark 5. *Actually, we can obtain a coarse version of the lemma, by replacing the inductive requirement on κ^+ with the same requirement on H_{κ^+}. This removes the need to appeal to the* GCH *in (2).*

A computation now completes the proof of the lemma: Since

$$(\bar{\kappa_\alpha}^+)^{\mathcal{M}_\alpha} \subseteq \mathrm{Hull}^{\mathcal{M}_\alpha}(i^{\mathcal{T}}_{0\alpha}[\mathcal{M}] \cup \bar{\lambda_\alpha}),$$

we have that H can compute the size of $\bar\kappa_b$ to be at most $|\mathcal{M}|+|\bar\lambda_b| < \bar\kappa_b$. But then $\bar\kappa_b = \kappa$ is not a cardinal of H, contradiction. □

Finally, let κ be an $L(\mathbb{R})$ cardinal, and let $f \in \mathsf{HOD}_x$ for some real x. Assume first that $M^\sharp_\omega(x)$ exists and is ω_1+1 iterable in the extension of the universe by $\mathrm{Coll}(\omega,\mathbb{R})$. Let $\mathcal{M}$ be in the HOD_x system (see

[Ste10, Section 8] and [Ste96] for notation), with $\pi_{\mathcal{M},\infty} : \mathcal{M} \to \mathsf{HOD}|\Theta$, $\pi_{\mathcal{M},\infty}(f_{\mathcal{M}}) = f$, and $\pi_{\mathcal{M},\infty}(\kappa_{\mathcal{M}}) = \kappa$.

The map $\pi_{\mathcal{M},\infty}$ is an iteration map and the lemma would apply as long as $L(\pi_{\mathcal{M},\infty}[\mathcal{M}], \mathsf{HOD}|\Theta) \models \kappa$ is a cardinal. But, of course, $\pi_{\mathcal{M},\infty}[\mathcal{M}]$ may collapse κ. So we argue more carefully, and remove the appeal to hypothesis beyond determinacy.

More precisely, we have a map not with domain $\mathcal{M}$ but defined on the hull $H_{\mathcal{M},S}$ of $\mathcal{M}$, where we only need to consider sets $S = S_i = \{\mu_0, \dots, \mu_i\}$ where the μ_i are uniform $\mathbb{R}$-indiscernibles (so we can dispense with the assumption that $M^\sharp_\omega(x)$ exists).

The point is that the image $\pi_{\mathcal{M},S_i}[H_{\mathcal{M},S_i}]$ *is* in $L(\mathbb{R})$, so we can apply the lemma in $L(\pi_{\mathcal{M},S_i}[H_{\mathcal{M},S_i}], \mathsf{HOD}|\Theta)$, a model where κ is a cardinal (since it is a cardinal of $L(\mathbb{R})$). This completes the proof. □

3. Remarks

It is clear from the proof that the result holds not just in $L(\mathbb{R})$ but in any AD^+ model of the form $L(\mathcal{P}(\mathbb{R}))$ where the mouse set theorem holds and we have an appropriate version of the analysis of HOD as the limit of an appropriate directed system.

In private communication, Sargsyan has pointed out that one actually does not need the mouse set theorem, or the HOD analysis, but only a coarse version of it. For instance, to prove that κ is Jónsson, all we need to run the argument is an iteration strategy Σ with Wadge rank above κ, and then we can use the M^Σ_ω-direct limit to prove the result as sketched above.

So the result is just a theorem of AD^+. The question remains of whether AD suffices to prove that all cardinals below Θ are Jónsson.

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