On Kechris Conjecture

Based on talks by John Clemens

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The underlying metatheory here includes AD, although it can be eliminated from most of the discussion when we restrict our attention to Borel functions.

Turing equivalence $\equiv_T$ is arithmetic (thus Borel) and countable (i.e., every class is.)

Definition 1. E is *Universal* among Countable Borel Equivalence Relations (CBER) iff

1. E is a CBER.
2. For any CBER F, $F \leq_B E$ (i.e., $\exists f$ Borel $(xFy \Leftrightarrow f(x)Ef(y))$.)

Fact 2 ($\mathsf{ZF}+\mathsf{DC}$)**.** *There is a universal CBER E_∞.* $\square$

Up to bireducibility, there is a unique such equivalence relation.

Example 3. *1. $X = 2^{\mathbb{F}_2}$, $\mathbb{F}_2$ = free group on 2 generators.*

$\mathbb{F}_2$ acts on X by shifting: For $\alpha \in 2^{\mathbb{F}_2}$ and $g \in \mathbb{F}_2$,

$$(g \cdot \alpha)(h) = \alpha(g^{-1}h).$$

2. $\overset{=\omega}{\sim}_{rec}$ on ω^ω: $x \overset{=\omega}{\sim}_{rec} y$ iff

$$\exists \pi \text{ recursive} \left(\pi \in S_\infty \text{ and } \forall n\, (x(n) = y(\pi(n)))\right).$$

Question 4. *Similarly, define $\overset{=2}{\sim}_{rec}$ on 2^ω. Is $\overset{=2}{\sim}_{rec}$ universal?*

Theorem 5 (Slaman, Steel. Unpublished). *$\equiv_{Arithmetic}$ is universal.* $\square$

Question 6. *Is $\equiv_T$ universal?*

Conjecture 7 (Kechris). *If E is universal, F countable and $E \subseteq F$ (literally, i.e., every E-class is contained in a single F-class), then F is also universal.*

Notice:

1. $\overset{=\omega}{\sim}_{\text{rec}} \subseteq \equiv_T$ (thinking of $\equiv_T$ on ω^ω.)

2. $\equiv_T \leq_B \overset{=2}{\sim}_{\text{rec}}$ (thinking of $\equiv_T$ on 2^ω.)

   This is because $x \equiv_T y \Leftrightarrow x' \equiv_1 y'$, so $f(x) = x'$ witnesses the reduction.

3. "F is universal" is a Borel (hence absolute) statement.

Assume $\equiv_T$ is universal. Here are some consequences of this assumption:

Let $\mathcal{D}$ be the structure of the Turing degrees. Let E be the equivalence relation $\equiv_T \times \Delta(2)$ on $2^\omega \times 2$: $(x_1, i_1)E(x_2, i_2)$ iff $i_1 = i_2$ and $x_1 \equiv_T x_2$. By universality, $E \leq_B \equiv_T$, so there is a function $f : \mathcal{D} \times 2 \to \mathcal{D}$ witnessing such a reduction. f induces 2 functions $f_0, f_1 : \mathcal{D} \to \mathcal{D}$. Notice that f_0 and f_1 are $\equiv_T$-invariant,

$$\forall x, y\, (f_0(x) \not\equiv_T f_1(y))$$

and $\emptyset = (f_0 \text{``}\omega) \cap (f_1 \text{``}\omega)$. Let $X_i = f_i \text{``}\omega$.

Either $X_0 \cup X_1$ contains a cone of degrees, or it does not, and even if it does, then at least one of them must miss a cone anyway. Now, if $\varphi : \mathcal{D} \to \mathcal{D}$ is injective and $\varphi \text{``}\mathcal{D}$ does not contain a cone, then $\varphi$ is not increasing (or, since $\varphi \text{``}\mathcal{D}$ would be cofinal, it would contain a cone) and φ is not constant on a cone (being injective).

Hence, universality of $\equiv_T$ contradicts Martin's conjecture!

Fact 8. *Suppose $E \leq_B F \leq_B E$ are both compressible CBER. Then $E \cong_B F$ (i.e., there is a Borel bijection of the spaces witnessing the bireducibility).* □

Fact 9. *$\equiv_T, E$ are compressible.* □

So we would have $\equiv_T \times \Delta(2) \cong \equiv_T$. Let $f : \mathcal{D} \times 2 \to \mathcal{D}$ be such an isomorphism. Let $A = f \text{``}\mathcal{D} \times \{0\}$ and $B = f \text{``}\mathcal{D} \times \{1\}$. Then they are disjoint, and we can induce an involution $g : A \to B$ (i.e., $g^2 =$id.) A and B are Borel sets, and their union is $\mathcal{D}$. Notice that if one of them contains a cone, the other cannot. This g is thus sidewise!

Definition 10. g is sidewise iff $g(x) \perp_T x$ a.e.

The reason for this is that $g(x) \geq_T x$ a.e. implies $x = g^2(x) \geq_T g(x) \geq_T x$ a.e., so $g(x) \equiv_T x$ a.e. Similarly, if $g(x) \leq_T x$ a.e., then either $g(x) \equiv_T x$ a.e., or else $g(x) <_T x$ a.e. (and hence g is constant a.e., by results of Slaman and Steel.)

Remark 11. $\equiv_{\text{Arithmetic}}$ is compressible, so such f exists for $\equiv_{\text{Arithmetic}}$.

Why is $\equiv_{\text{Arithmetic}}$ universal?

Uniformly $\equiv_{\text{Arithmetic}}$-invariant functions can be "ugly": Notice, e.g., that we can diagonalize $(x \mapsto (x^{(n)})_n)$ and get the real $(0^{(n)})_n$, which is not arithmetic, while with $\equiv_T$, we we "diagonalize" this phenomenon cannot occur. Also, non-uniformly Turing-invariant functions are hard to build.

Question 12. (AD) *Is there $f : 2^\omega \to 2^\omega$ such that $f(x) \equiv_T x'$ a.e., but f is not uniformly degree-invariant a.e.?*

The "natural" first way to try would be to use a priority argument. But these constructions tend not to be degree-invariant. It is open whether there is a degree-invariant solution to Post problem.

The question is still open even if instead of the AD-context we restrict our attention to Borel functions.

Slaman has pointed out that it is not even known if there are r.e. operators which solve Post problem on the Δ^0_3-sets. The idea here would be to look for larger and larger ideals where such a solution could be found. If on a small ideal the construction were to fail, we would have saved ourselves the trouble of constructing a fairly elaborate priority argument.