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On guessing generalized clubs at the successors of regulars. (English summary)
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This nice paper is an addition to the literature on diamond principles and other guessing principles, and in particular to how they relate to the construction of Souslin trees and to saturation properties of the nonstationary ideal [see also A. Rinot, in *Set theory and its applications*, 125–156, Contemp. Math., 533, Amer. Math. Soc., Providence, RI, 2011; MR2777747].

More specifically, the paper continues a line of investigations started in [B. König, P. B. Larson and Y. Yoshinobu, Fund. Math. **195** (2007), no. 2, 177–191; MR2320769 (2008f:03064)], where principles for guessing generalized clubs were considered. In that paper, the strong generalized club principle $⅄^*(\kappa, S)$ was introduced. Suppose that $\kappa \leq \lambda$, and that S is a stationary subset of the regular uncountable cardinal λ. Then $⅄^*(\kappa, S)$ asserts that there is a sequence

$$(C_\delta \mid \delta \in S)$$

such that:

(1) for every $\delta \in S$, C_δ is club in $[\delta]^{<\kappa}$, and
(2) for every club D in $[\lambda]^{<\kappa}$, there exists a club $C \subseteq \lambda$ such that

$$S \cap C \subseteq \{\delta \in S \mid \exists x \in C\delta\,(x \subseteq y \in C_\delta \Rightarrow y \in D)\}.$$

It is then shown that if $\lambda = \lambda^{<\lambda}$ is regular and $2^\lambda = \lambda^+$, then $⅄^*(\lambda, S^{\lambda^+}_\lambda)$ implies the existence of a λ-closed λ^+-Souslin tree. This continues themes going back to Jensen's work on L [see R. B. Jensen, Ann. Math. Logic **4** (1972), 229–308; erratum, ibid. **4** (1972), 443; MR0309729 (46 #8834)].

Suppose that λ is regular and uncountable, and $T \subseteq \lambda$ and $S \subseteq S^{\lambda^+}_\lambda$ are stationary. Definition 2.2 introduces *reflected diamond*, $\langle T \rangle_S$, the assertion that there are sequences

$$(C_\delta \mid \delta \in S) \qquad \text{and} \qquad (A^\delta_i \mid \delta \in S, i < \lambda)$$

such that:

(1) for every $\delta \in S$, C_δ is a club subset of δ of type λ, whose increasing enumeration we denote by $(\delta_i \mid i < \lambda)$, and
(2) for every club $D \subseteq \lambda^+$ and every $A \subseteq \lambda^+$, there are stationarily many $\delta \in S$ such that $\{i \in T \mid \delta_{i+1} \in D \text{ and } A \cap \delta_{i+1} = A^\delta_{i+1}\}$ is stationary in λ.

This principle (a version of the *usual* club guessing) is interpolated between $⅄^*$ assumptions (which are assertions of *generalized* club guessing) and their consequences. For example (Definition 1.6), suppose that $\kappa \leq \lambda$, that λ is regular and uncountable, and that $S \subseteq \lambda$ is stationary. The principle $⅄^-(\kappa, S)$ asserts that there is a sequence

$$(C^i_\delta \mid \delta \in S, i < |\delta|)$$

such that:

(1) for every $\delta \in S$ and $i < |\delta|$, C^i_δ is cofinal in $[\delta]^{<\kappa}$, and
(2) for every club D in $[\lambda]^{<\kappa}$, the set

$$\{\delta \in S \mid \exists i < |\delta| \, (C^i_\delta \subseteq D)\}$$

is stationary.

It is clear that if $S \subseteq \lambda^+$, then $⅄^*(\lambda, S)$ implies $⅄^-(\lambda, S)$. Moreover, in Theorem 2.5 it is shown that if λ is regular and uncountable, and $T \subseteq \lambda$ and $S \subseteq S^{\lambda^+}_\lambda$ are stationary, then each of the following statements implies the next one:

(1) $\diamondsuit_S$.
(2) $2^\lambda = \lambda^+$, and $⅄^-(\kappa, S)$ holds for some $\kappa < \lambda$.
(3) $\langle T \rangle_S$.

If in addition the nonstationary ideal on λ restricted to T is saturated, then $\langle T \rangle_S$ implies $\diamondsuit_S$.

In Theorem 2.7, it is shown that if $\lambda = \lambda^{<\lambda}$ is uncountable, and $\langle T \rangle_{S^{\lambda^+}_\lambda}$, then there is a λ-complete λ^+-Souslin tree.

Additional results are presented, showing the usefulness of the new principles. Several questions are also stated. Particularly interesting in my opinion is whether GCH is consistent with the failure, for some regular uncountable λ, of the principles $\langle T \rangle_{S^{\lambda^+}_\lambda}$ for all stationary $T \subseteq \lambda$.

Reviewed by *Andrés Eduardo Caicedo*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.