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Cardinal characteristics and projective wellorders. (English summary)
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This paper is the first to establish the consistency of inequalities among cardinal characteristics of the continuum while in the presence of a simply definable well-ordering of the reals. Specifically, the authors prove that any of the following inequalities is consistent with a (lightface) Δ^1_3 well-ordering: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, and $\mathfrak{b} < \mathfrak{g}$. Here, $\mathfrak{c}$ is the size of the continuum, $\mathfrak{a}$ is the almost disjointness number, $\mathfrak{b}$ is the bounding number, $\mathfrak{g}$ is the groupwise density number, and $\mathfrak{s}$ is the splitting number [see A. Blass, in *Handbook of set theory*, 395–489, Springer, Dordrecht, 2010; Zbl 1198.03058].

In each instance, the result is obtained by forcing over L with an appropriate countable support iteration. The individual posets are S-proper for a suitable stationary co-stationary subset S of ω_1, and S. Shelah's results on S-properness are used in an essential way [see *Proper and improper forcing*, second edition, Perspect. Math. Logic, Springer, Berlin, 1998; MR1623206 (98m:03002)]. For example, the iteration has length ω_2 and makes $\mathfrak{c} = \omega_2$ while preserving ω_1. The set S is chosen so that it is almost disjoint from a given ω_2-sequence $\overline{S}$ of almost disjoint stationary subsets of ω_1. The Δ^1_3-well-ordering is obtained by coding a well-ordering of the reals definably from the sequence $\overline{S}$ (by preserving or destroying the stationarity of some of these sets appropriately), and also coding the resulting relation using reals by applying David's trick [see S.-D. Friedman, *Fine structure and class forcing*, de Gruyter Ser. Log. Appl., 3, de Gruyter, Berlin, 2000; MR1780138 (2001g:03001)].

The iteration gives the authors enough leeway to implement additional combinatorial properties of the individual posets in order to obtain the inequalities between cardinal characteristics. For example, by ensuring that the posets are ω^ω-bounding, the resulting extension satisfies $\omega_1 = \mathfrak{d} < \mathfrak{c} = \omega_2$. For another example, consider $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \omega_2$. That $\mathfrak{b} = \omega_1$ is obtained by ensuring that the posets involved are weakly bounding (so the ground model reals form an unbounded family). That $\mathfrak{s} = \omega_2$ is obtained by cofinally often along the iteration using a poset defined by S. Shelah [in *Axiomatic set theory (Boulder, Colo., 1983)*, 183–207, Contemp. Math., 31, Amer. Math. Soc., Providence, RI, 1984; MR0763901 (86b:03064)] that adds a real not split by ground model reals. Finally, that $\mathfrak{a} = \omega_2$ is ensured by cofinally often along the iteration destroying the maximality of a candidate mad family of size ω_1.

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References

1. U. Abraham, Proper Forcing, in: The Handbook of Set-Theory, 2010.
2. A. Blass, Combinatorial Cardinal Characteristics of the Continuum, in: The Handbook of Set-Theory, 2010.

3. A. Blass, Applications of superperfect forcing and its relatives, in: Set theory and its applications, in: Lecture Notes in Math., vol. 1401, Springer-Verlag, Berlin, 1989. MR1031763 (91b:03081)
4. A. Caicedo, S. Friedman, BPFA and Projective Well-orderings of the Reals, submitted.
5. R. David, A very absolute Π^1_2 real singleton, Ann. Math. Logic 23 (1982) 101–120. MR0701122 (84m:03057)
6. K. Devlin, Constructibility, in: Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1984. MR0750828 (85k:03001)
7. A. Dow, More set theory for topologists, Topology Appl. 64 (1995) 243–300. MR1342520 (97a:54005)
8. S. Friedman, Fine Structure and class forcing, in: de Gruyter Series in Logic and its Applications, vol. 3, Walter de Gruyter & Co., Berlin, 2000. MR1780138 (2001g:03001)
9. S. Friedman, Lecture notes on definable wellorders, http://www.logic.univie.ac.at/∼sdf/.
10. M. Goldstern, A taste of proper forcing, in: Set Theory (Curaçao, 1995; Barcelona, 1996), Kluwer Acad. Publ., Dordrecht, 1998, pp. 71–82. MR1601976 (99f:03069)
11. L. Harrington, Long projective wellorderings, Ann. Math. Logic 12 (1977) 1–24. MR0465866 (57 #5752)
12. T. Jech, Set theory, in: Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR1940513 (2004g:03071)
13. S. Shelah, Proper and Improper Forcing, second edition, in: Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998. MR1623206 (98m:03002)
14. S. Shelah, On cardinal invariants of the continuum, Contemp. Math. 31 (1984) 184–207. MR0763901 (86b:03064)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.