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The consistency strength of the tree property at the double successor of a measurable cardinal. (English summary)

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If κ is an infinite cardinal, a κ-tree is a tree of height κ all of whose levels have size strictly smaller than κ. The *tree property* holds at κ iff any κ-tree has a branch of length κ. A counterexample to the tree property is a κ-*Aronszajn tree*.

The question of which cardinals have the tree property is an active topic of research in set theory. See for example [M. D. Foreman, M. Magidor and R.-D. Schindler, J. Symbolic Logic **66** (2001), no. 4, 1837–1847; MR1877026 (2003m:03083)], or [D. Sinapova, "The tree property and the failure of the singular cardinal hypothesis at $\aleph_{\omega^2}$", preprint, available at www.math.uci.edu/~dsinapov].

The paper under review studies the consistency strength of the tree property at the double successor of a measurable cardinal, and the main result is that it is precisely the consistency of a *weakly compact hypermeasurable* cardinal, where κ is such a cardinal iff there is a weakly compact cardinal $\lambda > \kappa$ and an elementary embedding $j\colon V \to M$ of the universe into a transitive class with $\mathrm{crit}(j) = \kappa$ and $H(\lambda)^M = H(\lambda)$.

The paper includes a nice and detailed presentation of (generalized) Sacks forcing and its iterations, including a proof of a theorem of A. Kanamori [see Ann. Math. Logic **19** (1980), no. 1-2, 97–114; MR0593029 (82i:03061)] showing that if ρ is strongly inaccessible and λ is weakly compact then, after forcing with $\mathbb{S}$, we have $\lambda = \rho^{++} = 2^\rho$ and ρ^{++} has the tree property, where $\mathbb{S} = \mathrm{Sacks}(\rho, \lambda)$ is the λ-length iteration using supports of size at most ρ of the version of Sacks forcing that adds a subset of ρ.

The main theorem is established in two stages. First, if GCH holds and κ is weakly compact hypermeasurable, a poset is described that preserves the measurability of κ and forces the tree property at κ^{++}. The poset is an Easton support iteration of a lottery of iterated Sacks forcing posets as above [see J. D. Hamkins, Ann. Pure Appl. Logic **101** (2000), no. 2-3, 103–146; MR1736060 (2001i:03108)].

Second, building on results of M. Gitik [see Ann. Pure Appl. Logic **63** (1993), no. 3, 227–240; MR1237232 (94m:03086)], it is shown that if κ is measurable and the tree property holds at κ^{++}, then either $0^\P$ exists, or else in the core model K, the Mitchell ordering of κ is at least $\lambda + 1$, where $\lambda > \kappa$ is weakly compact in K. In either case, it immediately follows that there is an inner model where κ is weakly compact hypermeasurable.

Reviewed by *Andrés Eduardo Caicedo*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.