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Gitik, Moti (IL-TLAV)
On changing cofinality of partially ordered sets. (English summary)
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In the context of set theory, we say that W is an *outer model* of V iff $W \supseteq V$, they have the same ordinals, and both V and W are models of set theory. An outer model W is a *cofinality preserving extension* iff any regular cardinal in V is also regular in W. It is natural to wonder whether there is a partially ordered set $P \in V$ and a cofinality preserving extension W where the cofinality of P is different from its cofinality in V. This is Problem 5.8 on A. Miller's list of problems, and is originally due to Watson and Dow. Let us say that $P \in V$ is *cofinality changeable* if there is such W.

The paper examines this question. It is shown that there are no cofinality changeable posets if V satisfies $\forall\tau\,(2^\tau < \tau^{+\omega})$. Along the way, it is shown that if κ is the least size of a cofinality changeable poset, then κ is singular in V and there is a cofinality changeable poset of size and cofinality κ in V.

The paper also shows that a positive answer to the question is consistent in the context of large cardinals, and its large cardinal strength is identified. Specifically, it is shown that if the generalized continuum hypothesis, GCH, holds, and there are ω measurable cardinals, then there is a cofinality changeable P. The argument uses Prikry type forcing notions [see M. Gitik, in *Handbook of set theory*, 1351–1447, Springer, Dordrecht, 2010; Zbl 1198.03062].

Conversely, appealing to an appropriate version of the covering lemma [see W. J. Mitchell, in *Handbook of set theory*, 1497–1594, Springer, Dordrecht, 2010; Zbl 1198.03067], it is shown that if there is a cofinality changeable P, then there is an inner model with a measurable cardinal, and that if, moreover, the core model K exists and every measurable of K is regular in V, then in fact there are ω measurable cardinals in K.

This argument is presented in Proposition 2.1 and Remark 2.2. I found the presentation a bit too fast, so I am adding some details to Remark 2.2 here (due thanks to Ralf Schindler): Assume that W is a cofinality preserving extension of V, and that W does not have an inner model with ω measurable cardinals. It follows that, for example, *zero-pistol* $0^\P$ does not exist, so K^V and K^W exist, each has only finitely many measurable cardinals, and K^V is an iterate of K^W (as shown by Jensen). First, note that no measurable of K^W can be used infinitely often in the iteration from K^W to K^V: Otherwise, we can look at an ω-sequence of critical points coming from the iteration of one measure. Their supremum has cofinality ω in V by hypothesis, and it is regular in K^V. This implies that it is measurable in K^V (for example, by results of Mitchell and Cox). Assuming that every measurable of K^V is regular in V (as the author does in Remark 2.2), this is a contradiction. Since K^W has only finitely many measurable cardinals, it follows that the iteration leading to K^V has finite length, and this implies that we have full covering between V and W. This allows us to run the argument of Proposition 2.1 to conclude.

The paper also presents additional results, dealing with the problem of changing the *outer* *cofinality* of a subset A of a poset $(P, \leq)$ in a cofinality preserving extension. Here, the outer cofinality of A in P is defined as the smallest size of a subset S of P that covers A in the sense that for any $a \in A$ there is a $b \in P$ with $a \leq b$.

Reviewed by *Andrés Eduardo Caicedo*

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.