MR2656116 (Review) 03B30 (03E15 03E57 03F30)
Koellner, Peter (1-HRV-Q)
Strong logics of first and second order. (English summary)
Bull. Symbolic Logic **16** (2010), *no. 1,* 1–36.

The paper is a natural follow-up to ideas introduced in [J. A. Väänänen, Bull. Symbolic Logic **7** (2001), no. 4, 504–520; MR1867954 (2002m:03010)]. It begins with a study of common features of ω-logic and β-logic, two logics usually studied in the context of second-order arithmetic and set theory that extend first-order logic. Isolating these features leads to the notion of *strong logic*, which is the main concern of the paper. Many interesting results (some well known) are discussed along the way, for example, the partial order of *interpretability* of theories.

Abstractly, a strong logic $\models_\Phi$ (of first order) can be identified with a collection of pairs (T,φ), where T is a (recursively enumerable) theory and φ is a sentence. We write $T\models_\Phi\varphi$ to mean that (T,φ) is a pair in the logic. The statement $\Phi(x)$ describes a class of *test structures* that are used to decide which pairs satisfy this relation, so that $T\models_\Phi\varphi$ iff for all M such that $\Phi(M)$, if $M\models T$, then $M\models\varphi$. The class of test structures can be quite general, and include Boolean-valued models.

The key features that the paper focuses on are generic invariance and faithfulness. Both notions depend on the background theory of sets, which typically is assumed to contain large cardinals. Given an extension $\mathrm{ZFC}^{(+)}$ of ZFC, the logic $\models_\Phi$ is said to be *generically invariant in* $\mathrm{ZFC}^{(+)}$ when, for all T and φ, we have that, provably in $\mathrm{ZFC}^{(+)}$, $T\models_\Phi\varphi$ iff $V^{\mathbb{P}}\models$ "$T\models_\Phi\varphi$" for any forcing notion $\mathbb{P}$. We say that $\models_\Phi$ is *faithful in* $\mathrm{ZFC}^{(+)}$ when, for all T and φ, we have that, provably in $\mathrm{ZFC}^{(+)}$, $\varnothing\models_\Phi\varphi$ iff $V^{\mathbb{P}}\models\varphi$ for all forcing notions $\mathbb{P}$.

The study of these logics naturally requires the use of notions familiar to set theorists, such as universal Baireness, and cannot be separated completely from set-theoretic considerations, usually coming from the study of the large cardinal hierarchy. Ultimately, this leads to a new presentation of Woodin's Ω-logic [see W. H. Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, de Gruyter Ser. Log. Appl., 1, de Gruyter, Berlin, 1999; MR1713438 (2001e:03001)].

The study of the appropriate version of these notions for second-order logic supports, in the author's view, the claim in [W. V. O. Quine, *Philosophy of logic*, second edition, Harvard Univ. Press, Cambridge, MA, 1986; MR0844769 (87f:03016)] that second-order logic is "set theory in sheep's clothing". One of the key issues is that full second-order logic is generically fragile, even in the context of large cardinals.

Reviewed by *Andrés Eduardo Caicedo*

References

1. Joan Bagaria, Neus Castells, and Paul Larson, An Ω-logic primer, *Set theory* (Joan Bagaria and Stevo Todorcevic, editors), Trends in Mathematics, Birkh "auser, Basel, 2006, pp. 1–28.

MR2267144 (2007f:03076)

2. George Boolos, To be is to be a value of a variable (or to be some values of some variables), *The Journal of Philosophy*, vol. 81 (1984), no. 8, pp. 430– 449. MR0755478 (86f:03048)
3. , Nominalist platonism, *Philosophical Review*, vol. 94 (1985), no. 3, pp. 327–344.
4. Samuel Buss, *Bounded arithmetic*, Bibliopolis, Napoli, 1986. MR0880863 (89h:03104)
5. , First-order proof theory of arithmetic, In Handbook of Proof Theory [6], 1998, pp. 79–147. MR1640326 (2000b:03208)
6. Samuel Buss (editor), *Handbook of proof theory*, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1998. MR1640324 (99d:03051)
7. Rudolf Carnap, *Logische Syntax der Sprache*, Springer, Wien, 1934, translated by A. Smeaton as *The logical syntax of language*, Kegan Paul, London, 1937. MR0231696 (38 #24)
8. Solomon Feferman, Arithmetization of metamathematics in a general setting, *Funda- menta Mathematicae*, vol. 49 (1960), pp. 35–92. MR0147397 (26 #4913)
9. Qi Feng, Menacham Magidor, and W. Hugh Woodin, Universally Baire sets of reals, *Set theory of the continuum* (Haim Judah, Winfried Just, and W. Hugh Woodin, editors), Mathematical Sciences Research Institute, vol. 26, Springer-Verlag, Berlin, 1992, pp. 203– 242. MR1233821 (94g:03095)
10. Leon Henkin, Completeness in the theory of types, *The Journal of Symbolic Logic*, vol. 15 (1950), no. 2, pp. 81–91. MR0036188 (12,70b)
11. Joost Johannes Joosten, *Interpretability formalized*, Ph.D. thesis, Universiteit Utrecht, 2004.
12. Akihiro Kanamori, *The higher infinite: Large cardinals in set theory from their beginnings*, second ed., Springer Monographs in Mathematics, Springer, Berlin, 2003. MR1994835 (2004f:03092)
13. Peter Koellner, On the question of absolute undecidability, *Philosophia Mathemat- ica*, vol. 14 (2006), no. 2, pp. 153–188. MR2245398 (2007d:03083)
14. , Truth in mathematics: The question of pluralism, *New waves in philosophy of mathematics* (Ot ’avio Bueno and Øystein Linnebo, editors), Palgrave Macmillan, 2009, forthcoming.
15. Peter Koellner and W. Hugh Woodin, Incompatible Ω-complete theories, *The Journal of Symbolic Logic*, (2009), forthcoming. cf. MR2583814
16. , Large cardinals from determinacy, *Handbook of set theory* (Matthew Foreman and Akihiro Kanamori, editors), Springer, 2009, forthcoming.
17. Paul B. Larson, *The stationary tower: Notes on a course by W. Hugh Woodin*, University Lecture Series, vol. 32, American Mathematical Society, 2004. MR2069032 (2005e:03001)
18. Richard Laver, Certain very large cardinals are not created in small forcing extensions, *Annals of Pure and Applied Logic*, vol. 149 (2007), no. 1–3, pp. 1–6. MR2364192 (2009e:03099)
19. Per Lindstr “om, A note on weak second order logic with variables for elementarily definable relations, *The proceedings of the Bertrand Russel memorial conference* (*Uldum, 1971*), 1973, pp. 221–233. MR0351750 (50 #4238)
20. , *Aspects of incompleteness*, second ed., Lecture Notes in Logic, vol. 10, Association for Symbolic Logic, 2003. MR2014250 (2004i:03095)
21. Øystein Linnebo, Plural quantification exposed, *No ûs*, vol. 37 (2003), no. 1, pp. 71– 92. MR1953582 (2003j:03008)

22. Donald A. Martin, Multiple universes of sets and indeterminate truth values, *Topoi*, vol. 20 (2001), no. 1, pp. 5–16. MR1836985 (2002e:03017)
23. Donald A. Martin and Robert M. Solovay, A basis theorem for P1 3 sets of reals, *Annals of Mathematics*, vol. 89 (1969), pp. 138–159. MR0255391 (41 #53)
24. Franco Montagna and Antonella Mancini, A minimal predicative set theory, *Notre Dame Journal of Formal Logic*, vol. 35 (1994), no. 2, pp. 186–203. MR1295558 (95h:03117)
25. Jan Mycielski, Pavel Pudl ’ak, and Alan S. Stern, A lattice of chapters of mathe- matics (interpretations between theorems), *Memoirs of the American Mathematical Society*, vol. 84 (1990), no. 426, iv+70. MR0997901 (90g:03009)
26. Edward Nelson, *Predicative arithmetic*, Princeton Mathematical Notes, no. 32, Princeton University Press, 1986. MR0869999 (88c:03061)
27. Charles Parsons, The uniqueness of the natural numbers, *Iyyun*, vol. 39 (1990), pp. 13– 44.
28. , *Mathematical thought and its objects*, Cambridge University Press, 2008. MR2381345 (2009a:00005)
29. Willard V. Quine, Ontological relativity, *The Journal of Philosophy*, vol. 65 (1968), no. 7, pp. 185–212.
30. , *Philosophy of logic*, second ed., Harvard University Press, Cambridge, MA, 1986. MR0844769 (87f:03016)
31. Joseph R. Shoenfield, The problem of predicativity, *Essays on the foundations of mathematics* (Yehoshua Bar-Hillel, E. I. J. Poznanski, Michael O. Rabin, and Abraham Robinson, editors), Magnes Press, Hebrew University, Jerusalem, 1961, pp. 132–139. MR0164886 (29 #2177)
32. Wanda Szmielew and Alfred Tarski, Mutual interpretability of some essentially undecidable theories, *Proceedings of the International Congress of Mathematicians*, vol. 1, Cambridge, 1950, p. 734.
33. Jouko V “a “an “anen, Second-order logic and foundations of mathematics, this Bulletin, vol. 7 (2001), no. 4, pp. 504–520. MR1867954 (2002m:03010)
34. Albert Visser, An overview of interpretability logic, *Advances in modal logic*, CSLI Lecture Notes, vol. 87, CSLI Publishing, Stanford, CA, 1998, vol. 1, pp. 307–359. MR1688529
35. , Pairs, sets, and sequences in first-order theories, *Archive for Mathematical Logic*, vol. 47 (2008), pp. 299–326. MR2420729 (2009d:03147)
36. W. Hugh Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, de Gruyter Series in Logic and its Applications, vol. 1, de Gruyter, Berlin, 1999. MR1713438 (2001e:03001)
37. , The continuum hypothesis, *Logic Collouqium 2000* (Rene Cori, Alexander Razborov, Steve Todor ĉevi’c, and Carol Wood, editors), Lecture Notes in Logic, vol. 19, Association for Symbolic Logic, 2005, pp. 143–197. MR2143878 (2006m:03079)
38. , The continuum hypothesis, the generic-multiverse of sets, and the Ω-conjecture, preprint, 2006.

Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.