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More on regular and decomposable ultrafilters in ZFC. (English summary)
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The paper under review is part of the author's systematic study of regular ultrafilters. Recall that a (λ,μ)-*regular* ultrafilter D is one where there is a λ-sequence of members of D such that any μ of them have empty intersection. This notion is due to H. J. Keisler [see Bull. Amer. Math. Soc. **70** (1964), 644–647; MR0166106 (29 #3384)], and has been widely studied for its connections with model theory and set theoretic topology. Its study quickly leads to pcf theory and to relations with large cardinals; see the carefully compiled list of references in the paper.

The results in the paper hold under ZFC. In particular, they do not deal with consistency results obtained under additional large cardinal or fine structural assumptions. A common theme is the relation between degrees of regularity and decomposability of ultrafilters. Recall that an ultrafilter D over a set I is λ-*decomposable* iff I can be partitioned into λ pieces in such a way that any union of fewer than λ pieces is not in D. Many results are obtained, the literature in the subject is recalled, and additional problems and conjectures are stated. What follows is a small sample of results.

Theorem 4.3.a$'$ is the statement that if $m\ge 1$ and D is $(\beth_m(\kappa^{+n}),\beth_m(\kappa^{+m}))$-regular, then it is μ-decomposable for some $\mu\in[\kappa,2^\kappa]$. The connection goes through extensions of classical results that relate regularity of ultrafilters to the cardinality of ultrapowers. Specifically, if the ultrapower by D of 2^{2^κ} has size strictly larger than 2^{2^κ}, then D is μ-decomposable for some $\mu\in[\kappa,2^\kappa]$. The way that regularity enters the picture is through Proposition 4.1:

$$\left|\prod_D \nu^\kappa\right| \le \left|\prod_D \nu^{<\kappa}\right|^{\operatorname{cf}(\prod_D(k,<))}$$

for any D and any cardinals ν and κ, but the regularity of D affects the cofinality of its ultrapowers. For example, it is classic that if λ is regular and D is (λ,κ)-regular, then $\operatorname{cf}(\prod_D(\lambda,<))>\kappa$.

Many results of the paper concern the *transference* of regularity. For example, Corollary 6.4 shows that if $\kappa>\lambda$, D is (λ^+,κ)-regular, κ is regular, and λ is singular, then D is either (λ,κ)-regular, or (ω,κ')-regular for any $\kappa'<\kappa$. This and additional transference results in the presence of singular cardinals are obtained by extending arguments of Kanamori and Ketonen dealing with least functions [see A. Kanamori, Trans. Amer. Math. Soc. **220** (1976), 393–399; MR0480041 (58 #240)]. Recall that if D is an ultrafilter over I, a function $f\colon I\to\kappa$ is κ-*least* for D iff:

(1) $\{i\in I\mid \alpha<f(i)\}\in D$ for any $\alpha<\kappa$, and
(2) for any $g\colon I\to\kappa$, if $\{i\in I\mid g(i)<f(i)\}\in D$, then there is an $\alpha<\kappa$ such that $\{i\in I\mid g(i)<\alpha\}\in D$.

Recall that if $\mu \leq \lambda' \leq \lambda$, then the covering number $\mathrm{cov}(\lambda, \lambda', \mu)$ denotes the smallest size of a family of subsets of λ, each of size strictly less than λ', such that any subset of λ of size less than μ is contained in at least one set in the family. In Theorem 5.1 it is shown that if λ is limit and D is ν-decomposable for arbitrarily large $\nu < \lambda$, then it is κ-decomposable for some $\kappa \in [\lambda, \lambda^{\mathrm{cf}(\lambda)}]$. If in addition λ is singular and D is $(\mu, \mathrm{cf}(\lambda))$-regular then, for every $\lambda' \in [\mu, \lambda)$, there is a $\kappa \in [\lambda, \mathrm{cov}(\lambda, \lambda', \mu)]$ such that D is κ-decomposable.

There are additional results relating regularity of product ultrafilters to the regularity of their factors, and applications in abstract model theory, and set theoretic topology.

Reviewed by *Andrés Eduardo Caicedo*

References

1. Y. Abe and M. Shioya, Regularity of ultrafilters and fixed points of elementary embeddings. Tsukuba J. Math. **22,** 31–37 (1998). MR1637644 (99h:03018)
2. A. Adler and M. Jorgensen, Descendingly incomplete ultrafilters and the cardinality of ultrapowers. Can. J. Math. **24,** 830–834 (1972). MR0309724 (46 #8829)
3. A. W. Apter, J. Hamkins, and D. Joel, Exactly controlling the non-supercompact strongly compact cardinals. J. Symbolic Logic **68,** 669–688 (2003). MR1976597 (2004b:03075)
4. A. W. Apter and J. M. Henle, On box, weak box and strong compactness. Bull. London Math. Soc. **24,** 513–518 (1992). MR1183305 (94a:03079)
5. J. Adámek, V. Koubek, and V. Trnková, How large are left exact functors? Theory Appl. Categ. **8,** 377–390 (2001). MR1847037 (2002h:18002)
6. J. Barwise and S. Feferman (eds.), Model-theoretic Logics. Perspectives in Mathematical Logic (Springer-Verlag, 1985). MR0819531 (87g:03033)
7. J. L. Bell and A. B. Slomson, Models and Ultraproducts: an Introduction (North-Holland Publishing Co., 1969). MR0269486 (42 #4381)
8. S. Ben-David and M. Magidor, The weak $\square^*$ is really weaker than the full $\square$. J. Symbolic Logic **51,** 1029–1033 (1986). MR0865928 (88a:03117)
9. M. Benda and J. Ketonen, Regularity of ultrafilters. Israel J. Mathematics **17,** 231–240 (1974). MR0396264 (53 #132)
10. X. Caicedo, The abstract compactness theorem revisited. In: Logic and foundations of mathematics. Selected papers from the 10th International Congress of Logic, Methodology and Philosophy of Science held in Florence, August 1995. Synthese Library, Vol. 280 (A. Cantini, E. Casari, and P. Minari, eds.), pp. 131–141 (Kluwer Academic Publishers Group, 1999). MR1739865 (2001b:03039)
11. C. C. Chang, Descendingly incomplete ultrafilters. Trans. Amer. Math. Soc. **126,** 108–111 (1967). MR0214474 (35 #5324)
12. C. C. Chang and J. Keisler, Model Theory. Studies in Logic and the Foundations of Mathematics, Vol. 73, third edition (North-Holland Publishing Co., 1990). MR1059055 (91c:03026)
13. W. Comfort and S. Negrepontis, The Theory of Ultrafilters. Die Grundlehren der mathematischen Wissenschaften, Vol. 211 (Springer-Verlag, 1974). MR0396267 (53 #135)
14. G. V. Čudnovskiĭ and D. V. Čudnovskiĭ, Regular and descending incomplete ultrafilters (English translation). Soviet Math. Dokl. **12,** 901–905 (1971).

15. O. Deiser and H.-D. Donder, Canonical functions, non-regular ultrafilters and Ulam's problem on ω_1. J. Symbolic Logic **68,** 713–739 (2003). MR2000073 (2004k:03095)
16. M. Di Nasso, Hyperordinals and nonstandard α-models. In: Logic and algebra, Papers from the International Conference in memory of Roberto Magari held in Pontignano, April 26–30, 1994. Lecture Notes in Pure and Applied Mathematics, Vol. 180 (A. Ursini and P. Aglianò, eds.), pp. 457–475 (Marcel Dekker, Inc., 1996). MR1404952 (97i:03033)
17. M. Di Nasso and M. Forti, Hausdorff ultrafilters. Proc. Amer. Math. Soc. **134,** 1809–1818 (2006). MR2207497 (2006k:03084)
18. H.-D. Donder, Regularity of ultrafilters and the core model. Israel J. Mathematics **63,** 289–322 (1988). MR0969944 (90a:03071)
19. H.-D. Donder, R. B. Jensen, and B. J. Koppelberg, Some applications of the core model. In: Set Theory and Model Theory. Proceedings of a Symposium held in Bonn, June 1–3, 1979. Lecture Notes in Mathematics, Vol. 872 (R. B. Jensen and A. Prestel, eds.), pp. 55–97 (Springer-Verlag, 1981). MR0645907 (83c:03032)
20. H.-D. Ebbinghaus, Extended logics: the general framework. In: Model-theoretic Logics, Perspectives in Mathematical Logic (J. Barwise and S. Feferman, eds.), Chap. II (Springer-Verlag, 1985). MR0819533
21. M. Foreman, An $\aleph_1$-dense ideal on $\aleph_2$. Israel J. Mathematics **108,** 253–290 (1998). MR1669368 (2000e:03140)
22. M. Foreman, M. Magidor, and S. Shelah, Martin's maximum, saturated ideals and non-regular ultrafilters. Part II. Annals of Mathematics **127,** 521–545, (1988). MR0942519 (90a:03077)
23. S. Garcia Ferreira, On two generalizations of pseudocompactness. Topology Proc. **24** Proceedings of the 14th Summer Conference on General Topology and its Applications. Held at Long Island University, Brookville, NY, August 4–8, 1999, pp. 149–172 (2001). MR1876373 (2002k:54026)
24. A. Hajnal, Ulam-matrices for inaccessible cardinals. Bulletin de l'Academie Polonaise des Sciences Series de Sciences Math. Astr. Phys. **XVII,** 683–688 (1969). MR0260597 (41 #5221)
25. K. Hrbacek, Realism, nonstandard settheory, and large cardinals. Annals Pure Appl. Logic **109,** 15–48 (2001). MR1835236 (2002f:03127)
26. M. Huberich, Non-regular ultrafilters. Israel J. Math. **87,** 275–288 (1994). MR1286831 (95k:03085)
27. T. Jech, Set Theory. The third millennium edition, revised and expanded, Springer Monographs in Mathematics (Springer-Verlag, 2003). MR1940513 (2004g:03071)
28. T. Jech and K. Prikry, Ideals over uncountable sets: application of almost disjoint functions and generic ultrapowers. Memoirs Amer. Math. Soc. **18,** no. 214 (1979). MR0519927 (80f:03059)
29. M. Jorgensen, Regular ultrafilters and long ultrapowers. Canad. Math. Bull. **18,** 41–43 (1975). MR0392557 (52 #13374)
30. A. Kanamori, Weakly normal filters and irregular ultrafilters. Trans. Amer. Math. Soc. **220,** 393–399 (1976). MR0480041 (58 #240)
31. A. Kanamori, Finest partitions for ultrafilters. J. Symbolic Logic **51,** 327–332 (1986). MR0840409 (87f:04004)
32. A. Kanamori, The Higher Infinite. Large Cardinals in Set Theory from their Beginnings.

Springer Monographs in Mathematics, second edition (Springer-Verlag, 2003). MR1994835 (2004f:03092)

33. A. Kanamori and M. Magidor, The evolution of large cardinal axioms in set theory. In: Higher Set Theory. Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977. Lecture Notes in Math. Vol. 669 (G. H. Müller and D. S. Scott, eds.), pp. 99–275 (Springer, Berlin, 1978). MR0520190 (80b:03083)
34. J. Kennedy, S. Shelah, and J. Väänänen, Regular ultrafilters and finite square principles. J. Symbolic Logic **73,** 817–823 (2008). MR2444269 (2009h:03040)
35. J. Keisler, On cardinalities of ultraproducts. Bull. Amer. Math. Soc. **70,** 644–647 (1964). MR0166106 (29 #3384)
36. J. Ketonen, Strong compactness and other cardinal sins. Annals Math. Logic **5,** 47–76 (1972). MR0469768 (57 #9549)
37. J. Ketonen, Nonregular ultrafilters and large cardinals. Trans. Amer. Math. Soc. **224,** 61–73 (1976). MR0419236 (54 #7260)
38. J. Ketonen, Some combinatorial properties of ultrafilters. Fundamenta Mathematica **107,** 225–235 (1980). MR0585552 (81i:03080)
39. P. Komjáth, A second category set with only first category functions. Proc. Amer. Math. Soc. **112,** 1129–1136 (1991). MR1065086 (91j:03067)
40. K. Kunen, Locally compact linearly Lindelöf spaces. Comment. Math. Univ. Carolin. **43,** 155–158 (2002). MR1903314 (2003d:54040)
41. K. Kunen and K. L. Prikry, On descendingly incomplete ultrafilters. J. Symbolic Logic **36,** 650–652 (1971). MR0302441 (46 #1585)
42. K. Kunen and J. E. Vaughan (eds.), Handbook of Set-theoretic Topology (North-Holland Publishing Co., 1984). MR0776619 (85k:54001)
43. P. Lipparini, Limit ultrapowers and abstract logics. J. Symbolic Logic **52,** 437–454 (1987). MR0890451 (88g:03054)
44. P. Lipparini, About some generalizations of (λ, μ)-compactness. In: Proceedings of the 5*th* Easter Conference on Model Theory, Wendisch Rietz, 1987. Seminarber. Sekt. Math. **93,** pp. 139–141 (Humboldt-Universität Berlin, 1987). Available also at the author's web page http://www.mat.uniroma2.it/∼lipparin/art/easter85.pdf.
45. P. Lipparini, Consequences of compactness properties for abstract logics. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **80,** 501–503 (1987). MR0976942 (90d:03071)
46. P. Lipparini, The compactness spectrum of abstract logics, large cardinals and combinatorial principles. Boll. Un. Mat. Ital. **B** (7) **4,** 875–903 (1990). MR1086710 (93d:03043)
47. P. Lipparini, Compactness of cardinality logics and constructibility. In: Proceedings of the Tenth Easter Conference on Model Theory, Wendisch Rietz, Germany, April 12–17, 1993 (M. Weese et al., eds.), pp. 130–133 (Humboldt-Universität Berlin, Fachbereich Mathematik, 1993). MR1261444 (94i:03005)
48. P. Lipparini, Ultrafilter translations, I: (λ, λ)-compactness of logics with a cardinality quantifier. Arch. Math. Logic **35,** 63–87 (1996). MR1375069 (96k:03087)
49. P. Lipparini, Productive $[\lambda, \mu]$-compactness and regular ultrafilters. Topology Proc. **21,** 161–171 (1996). MR1489196 (98j:54043)

50. P. Lipparini, Every (λ^+, κ^+)-regular ultrafilter is (λ, κ)-regular. Proc. Amer. Math. Soc. **128,** 605–609 (1999). MR1623032 (2000c:03026)
51. P. Lipparini, Decomposable ultrafilters and cardinal arithmetic (abstract). Bulletin Symbolic Logic **5,** 281–282 (1999).
52. P. Lipparini, Regular ultrafilters and $[\lambda, \lambda]$-compact products of topological spaces (abstract). Bulletin Symbolic Logic **5,** 121 (1999).
53. P. Lipparini, Still more on regular ultrafilters (abstract). Bulletin Symbolic Logic **6,** 394 (2000).
54. P. Lipparini, A connection between decomposability of ultrafilters and possible cofinalities. http://arXiv.org/math.LO/0604191; II. http://arXiv.org/math.LO/0605022 (2006); III, in preparation.
55. P. Lipparini, Decomposable ultrafilters and possible cofinalities, Notre Dame J. Form. Log. **49,** 307–312 (2008). MR2428557 (2009e:03081)
56. P. Lipparini, Combinatorial and model-theoretical principles related to regularity of ultrafilters and compactness of topological spaces. I. http://arxiv.org/abs/0803.3498; II. http://arxiv.org/abs/0804.1445; III. http://arxiv.org/abs/0804.3737; IV. http://arxiv.org/abs/0805.1548 (2008); V. http://arxiv.org/abs/0903.4691; VI.http://arxiv.org/abs/0904.3104 (2009).
57. J. A. Makowsky, Compactness, embeddings and definability. In: Model-theoretic Logics. Perspectives in Mathematical Logic (J. Barwise and S. Feferman, eds.), Chap. XVII (Springer-Verlag, 1985). MR0819531 (87g:03033)
58. D. Peterson, Cardinal functions on ultraproducts of Boolean algebras. J. Symbolic Logic **62,** 43–59 (1997). MR1450513 (99e:03024)
59. K. Prikry, Changing measurable into accessible cardinal. Dissertationes Math. Rozprawy Mat. **LXVIII** (1970). MR0262075 (41 #6685)
60. K. Prikry, On descendingly complete ultrafilters. In: Cambridge Summer School in Mathematical Logic. Lecture Notes in Math. Vol. 337 (A. R. D. Mathias and H. Rogers, eds.), pp. 459–488 (Springer-Verlag, 1973). MR0347613 (50 #116)
61. K. Prikry, On the regularity of ultrafilters. In: Surveys in Set Theory. London Math. Soc. Lecture Note Ser., Vol. 87 (A. R. D. Mathias, ed.), pp. 162–166 (Cambridge Univ. Press, 1983). MR0823779 (87c:03096)
62. M. Sakai, Non-(ω, ω_1)-regular ultrafilter and perfect κ-normality of product spaces. Topology Appl. **45,** 165–172 (1992). MR1180807 (93g:54014)
63. E. Schimmerling and M. Zeman, Square in core models. Bulletin Symbolic Logic **7,** 305–314 (2001). MR1860606 (2002j:03053)
64. M. Sheard, Indecomposable ultrafilters over small large cardinals. J. Symbolic Logic **48,** 1000–1007 (1983). MR0727789 (85b:03091)
65. S. Shelah, Cardinal Arithmetic. Oxford Logic Guides, Vol. 29 (Oxford Science Publications. The Clarendon Press, Oxford University Press, 1994). MR1318912 (96e:03001)
66. S. Shelah, Applications of pcf theory. J. Symbolic Logic **65,** 1624–1674 (2000). MR1812172 (2002g:03096)
67. J. H. Silver, Indecomposable ultrafilters and $0^\#$. In: Proceedings of the Tarski Symposium. An international symposium held at the University of California, Berkeley, June 23–30, 1971, to honor Alfred Tarski on the occasion of his seventieth birthday. Proceedings of Symposia in

Pure Mathematics, Vol. XXV (Leon Henkin, John Addison, C. C. Chang et al., eds.), pp. 357–363 (Published for the Association for Symbolic Logic by the American Mathematical Society, Providence, R. I., 1974). MR0360276 (50 #12726)
68. J. R. Steel, PFA implies $\mathrm{AD}^{L(\mathbf{R})}$. J. Symbolic Logic **70,** 1255–1296 (2005). MR2194247 (2008b:03069)
69. A. D. Taylor, Regularity properties of ideals and ultrafilters. Annals Math. Logic **16,** 33–55 (1979). MR0530430 (83b:04003)
70. W. H. Woodin, The Axiom of Determinacy, Forcing axioms, and the Nonstationary Ideal. De Gruyter Series in Logic and its Applications, Vol. 1 (Walter de Gruyter & Co., 1999). MR1713438 (2001e:03001)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.