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Bagaria, Joan (E-BARU-HS); **Kanovei, Vladimir** (E-BARU-HS)
On coding uncountable sets by reals. **(English summary)**
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The paper under review is nicely written and carefully organized. It begins with a brief account of known coding results. The first theorem on coding an uncountable set of reals by a single real is due to R. B. Jensen and R. M. Solovay [in *Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968)*, 84–104, North-Holland, Amsterdam, 1970; MR0289291 (44 #6482)]. Their method of *almost disjoint coding* is carried out in two stages: Identify the set of reals with a set $A \subseteq \omega_1$. First, A is *reshaped*: A club $C \subseteq \omega_1$ is added so that the reals of $L[A][C]$ are the same as the reals of $L[A]$, and $\xi < \omega_1^{L[A\cap\xi]}$ for all ordinals $\xi \in C$. Then almost-disjoint forcing adds a real x to $L[A][C]$ such that $A, C \in L[x]$, so $L[A][C][x] = L[x]$.

This result was significantly extended by Jensen [see A. Beller, R. B. Jensen and P. Welch, *Coding the universe*, London Mathematical Society Lecture Note Series, 47, Cambridge Univ. Press, Cambridge, 1982; MR0645538 (84b:03002)], who showed that if $V = L[A]$ for some $A \subseteq$ ORD and GCH holds, then there is a cofinality preserving class forcing extension that has the form $L[x]$ for some real x, and where A is definable from r. It was further refined by S.-D. Friedman [Ann. Pure Appl. Logic **41** (1989), no. 3, 233–297; MR0984629 (90i:03056)], who showed that, in addition, the extension can be assumed *minimal* in the sense that for any $B \subseteq$ ORD in $L[x]$, either $B \in V$ or else $L[B] = L[x]$. Both of these arguments require a thorough understanding of fine structure.

The main new result of the paper is that if $A \subseteq \omega_1$ then there is a much simpler minimal coding. Specifically, assume that $A \subseteq \omega_1$ and $V = L[A]$. A real x is added by a certain forcing consisting of perfect trees (a subforcing of Sacks forcing), such that:

(1) There is in $L[x]$ a club $C \subseteq \omega_1$ that reshapes A.
(2) The set A is in $L[x]$, so $L[A][x] = L[x]$ and, in $L[x]$, A is Δ_1 definable over $H(\omega_1)$ from x.
(3) The real x is minimal, so that for any $Y \in V[x]$, either $x \in V[Y]$ or $Y \in V$.

The authors also show that Sacks forcing itself is in general not enough to achieve the result, and include a brief survey of similar negative results for a diverse class of posets. Building on results of R.-D. Schindler [J. Symbolic Logic **66** (2001), no. 3, 1481–1492; MR1856755 (2002g:03111); MLQ Math. Log. Q. **50** (2004), no. 6, 527–532; MR2096166 (2005g:03076)], they show that if ω_1^V is not remarkable in L, then item (2) of the main result together with (3) restricted to $Y \subseteq \omega$ can be achieved by proper forcing. Properness, however, prevents item (1) from holding in general. As a corollary, the existence of a remarkable cardinal is equiconsistent with the existence of an $A \subseteq \omega_1$ such that in $L[A]$ there is no semiproper forcing notion that codes A by a real. Similar equiconsistency results are established for other classes of forcing notions.

Reviewed by *Andrés Eduardo Caicedo*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.