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Fuchs, Gunter [Fuchs, Gunter[2]] (1-CUNYS)
Generic embeddings associated to an indestructibly weakly compact cardinal. (English summary)
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A weakly compact cardinal is indestructible iff it remains weakly compact after any $<\kappa$-closed forcing. This notion was first studied by A. W. Apter and J. D. Hamkins in [MLQ Math. Log. Q. **47** (2001), no. 4, 563–571; MR1865776 (2003h:03078)], where they proved that if the indestructibility of κ is forced by a poset with a closure point below κ, then κ was originally supercompact. (Recall that δ is a closure point of $\mathbb{P}$ iff we can factor $\mathbb{P}=\mathbb{Q}*\dot{\mathbb{R}}$ where $|\mathbb{Q}|\leq\delta$ and $\Vdash_{\mathbb{Q}}$"$\dot{\mathbb{R}}$ is $\leq\delta$-strategically closed.") They conjectured that indestructible weak compactness is indeed equiconsistent with supercompactness.

In the current paper, the author shows that the indestructible weak compactness of κ implies that "external" supercompactness measures on several $\mathcal{P}_\kappa(\lambda)$ can be added by forcing (see Theorem 2.7). This ensures that if $\mu<\kappa$ is regular and uncountable, then the forcing axiom $\mathsf{M}A^{++}(<\mu\text{-closed})$ holds after forcing with $\mathrm{Col}(\mu,<\kappa)$ (see Theorems 3.8 and 3.11).

These forcing axioms are shown to have significant consequences. For example, Corollary 3.14 shows that $\mathsf{M}A^{+}(\sigma\text{-closed})$ implies, among others, the Singular Cardinal Hypothesis (SCH), the pre-saturation of the non-stationary ideal on ω_1, and reflection of stationary subsets of $\mathcal{P}_{\aleph_1}(X)$ to sets of size $\aleph_1$, for any uncountable X. In particular, SCH holds above an indestructible weakly compact κ. Moreover, Corollary 3.23 and Theorem 3.24 show that κ is countably completely ω_1-Jónsson, and that after forcing with $\mathrm{Col}(\omega_1,<\kappa)$, the Strong Chang Conjecture holds. Recall that this principle (due to S. Shelah; see Theorem 2.5 in Chapter XII of [*Proper and improper forcing*, second edition, Perspect. Math. Logic, Springer, Berlin, 1998; MR1623206 (98m:03002)]) states that for any structure $\mathfrak{A}=(A,\omega_1,\dots)$ of type (ω_2,ω_1) there is a club $C\subseteq\omega_1$ such that for any $\alpha\in C$ there is an elementary substructure $\mathfrak{B}\prec\mathfrak{A}$ of type (ω_1,ω) with $\mathfrak{B}\cap\omega_1=\alpha$.

The notion of a countably completely ω_1-Jónsson cardinal κ is a weakening of Woodin's notion of completely Jónsson cardinals [see P. B. Larson, *The stationary tower*, Univ. Lecture Ser., 32, Amer. Math. Soc., Providence, RI, 2004; MR2069032 (2005e:03001)]. This variant states that κ is inaccessible and that whenever $a\in V_\kappa$ is stationary and consists of countable sets, then there is a stationary set of $X\in V_\kappa$ such that $X\cap(\bigcup a)\in a$ and $X\cap\kappa$ is uncountable. This suffices to establish enough properties of the countable tower $\mathbb{Q}_{<\kappa}$ to prove Theorem 4.15: If κ is indestructibly weakly compact, then the Chang model $L(\mathrm{ORD}^\omega)$ elementary embeds in the Chang model of the extension by $\mathrm{Col}(\omega,<\kappa)$. In particular, all sets of reals in the model are Lebesgue measurable, have the Baire property, etc.

This substantial paper is very nicely written and suggests a promising line of research on this topic.

Reviewed by *Andrés Eduardo Caicedo*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.