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The two-cardinal problem for languages of arbitrary cardinality. (English summary)
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Given cardinals $\kappa \geq \lambda$ and $\mu \geq \nu$, the two-cardinal transfer theorem

$$(\kappa, \lambda) \Rightarrow (\mu, \nu)$$

holds for a language $\mathcal{L}$ iff there is a (distinguished) unary predicate $U \in \mathcal{L}$ such that, for any $\mathcal{L}$-theory T and any model $\mathcal{M} = (M, U^{\mathcal{M}}, \dots) \models T$ with $|M| = \kappa$ and $|U^{\mathcal{M}}| = \lambda$, there is a model $\mathcal{N} = (N, U^{\mathcal{N}}, \dots) \models T$ with $|N| = \mu$ and $|U^{\mathcal{N}}| = \nu$.

These theorems are of interest in classical model theory [see, for example, C. C. Chang and H. J. Keisler, *Model theory*, third edition, Stud. Logic Found. Math., 73, North-Holland, Amsterdam, 1990; MR1059055 (91c:03026) (Section 7.2)]. It has been clear for several decades that cardinal transfer results tend to require set theoretic machinery and additional assumptions beyond ZFC. Jensen's morasses turn out to be particularly useful [see, for example, K. J. Devlin, *Constructibility*, Perspect. Math. Logic, Springer, Berlin, 1984; MR0750828 (85k:03001) (Chapter VIII)].

In the paper under review, the author claims to prove that the two-cardinal transfer theorem

$$(\kappa^+, \kappa) \Rightarrow (\kappa^{++}, \kappa^+)$$

holds in the constructible universe L for any language $\mathcal{L}$. Note that we may assume that $\mathcal{L}$ has size at most κ^{++}. This is part of problem 7.2.17 in the book by Chang and Keisler.

A brief review of previous results is in order: If the language $\mathcal{L}$ is countable, [C. C. Chang, Proc. Amer. Math. Soc. **16** (1965), 1148–1155; MR0193016 (33 #1238)] proves the gap-1 transfer theorem

$$(\kappa^+, \kappa) \Rightarrow (\lambda^+, \lambda)$$

for regular λ assuming the Generalized Continuum Hypothesis, GCH. J. Silver extended the result to λ singular, assuming GCH and $\square_\lambda$ [see R. B. Jensen, Ann. Math. Logic **4** (1972), 229–308 (Section 7); erratum, ibid. **4** (1972), 443; MR0309729 (46 #8834)]. Recently, the author proved the gap-1 transfer theorem for any language $\mathcal{L}$ of size at most λ, assuming that λ is regular and there is a coarse $(\lambda, 1)$-morass [see MLQ Math. Log. Q. **52** (2006), no. 4, 340–350; MR2252965 (2007d:03086)]. It follows that, for the two-cardinal transfer theorem, only the case $|\mathcal{L}| = \kappa^{++}$ is pending.

Unfortunately, there are several problems with the paper under review. Work in L and fix a language $\mathcal{L} = \{U\} \cup \{R_\nu \mid \nu < \kappa^{++}\}$ and a structure $\mathfrak{A} = (L_{\kappa^+}, L_\kappa, B_\nu)_{\nu < \kappa^{++}}$. The paper observes that, in L, there is a coarse $(\kappa^+, 1)$-morass. This is not proved (the reference given is to unpublished notes by Jensen); note that in item (b) of Definition 4.1, "closed" should be replaced with "closed in μ^+". The omission of an explicit construction or available reference is unfortunate: For example, on page 793, lines 1–3, a function is defined and stated to be $\Sigma_1(\{\alpha_\nu\})$ but it is not clear what this means, since the function explicitly refers to the sets B_ν. This is problematic, as only Σ_1-

elementarity (in restricted languages) is guaranteed for several maps throughout the paper. It does not seem to be an essential problem, as the required elementarity can be built in from the beginning by a small modification of Jensen's original construction of the coarse morass, but this should have been made explicit by the author.

Sections 1–6 are preliminary. Section 5 is particularly problematic. Detailed arguments are given there for various immediate claims, and several mistakes are introduced. Note that a more streamlined presentation of precisely the same material is given in Section 4 of the paper by the author referred to above. In the current paper, Lemma 5.4 as stated is obviously false. A counterexample is given by taking $n = 1$, $\varphi(y, a, x) \equiv$ "x is the $\lhd$-successor of y", and $\psi(y, a, x) \equiv y = y$. The author has explained to me that the correct statement of the lemma requires that $x_1, \ldots, x_n$ be $\lhd$-bounded by some fixed w, which renders the lemma trivial. (Note, however, that the proof as written is incorrect.) A more serious problem is in the proof of Proposition 5.6 (Proposition 4.5 of the author's paper cited above). Note that starting in line -4 of this proof, a claim is made that there is a w (independent of v) that $\lhd$-bounds b, b', a and therefore allows us to quote Lemma 5.4. This is not justified and invalidates the argument. The author seems to be saying that since, for any fixed v there is such a bound, then a bound can be picked uniformly for cofinally many values of v (this is precisely the same mistake that appears in the proof of Lemma 5.4). The difference with the proof given in the author's paper cited above is that the variable denoted a' in that paper is implicitly bounded uniformly from the beginning as it must belong to the interpretation of the predicate U, which is $\lhd$-bounded.

Lemma 5.6 is used in an essential manner in the proof of the main result, in Section 7, specifically in the proof of Claim 2 on page 797. I do not see at the moment how to ensure that the relevant variables are bounded in order to make a valid appeal to the correct version of Lemma 5.4 and therefore cannot confirm the correctness of the proof as published. Let me add that the argument in Section 7 is very close to the one given in Section 5 of the author's paper cited above, replicating entire paragraphs, and it would perhaps have been more appropriate to publish a single paper rather than two.

Reviewed by *Andrés Eduardo Caicedo*

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.