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Woodin, W. Hugh (1-CA)

★**The realm of the infinite.**

Infinity, 89–118, *Cambridge Univ. Press*, *Cambridge*, 2011.

This paper discusses for a general mathematical and philosophically inclined audience the Inner Model Program and Woodin's recent contributions to it through what he has called the search for "ultimate L". Most of the technical details are left out, but references where they can be found are provided.

The framework of the discussion is whether one can coherently talk of the realm of the infinite, and it begins with the idea that the mathematical development of set theory leads to certain predictions "about the physical universe", for example, the assertion that an inconsistency cannot be found in the theory ZF + AD. Actually, Woodin begins by reviewing arguments he first presented in "The tower of Hanoi" [in *Truth in mathematics (Mussomeli, 1995)*, 329–351, Oxford Univ. Press, New York, 1998; MR1688350] that challenge a view of mathematics where the only meaningful existence assertions are those that can be verified in the physical universe. This view certainly excludes set theory (the mathematics of infinite sets). For this, Woodin explains how to produce an explicit statement that, if true, precludes the existence of $V_{|V_{1000}|}$ and yet (at least in principle) is physically feasible, and can be verified in fewer than 10^{48} steps. As Woodin indicates, we expect the statement to be false, but there does not seem to be a coherent argument that would explain why unless we accept as meaningful the conception of a nonphysical realm, a view where it makes sense to talk of the existence of at least some infinite objects. (The argument is presented here rather briskly; for a more leisurely presentation, see "The tower of Hanoi".)

Woodin proceeds by reviewing the argument of his paper "The transfinite universe" [in *Kurt Gödel and the foundations of mathematics: horizons of truth*, M. Baaz et al. (eds.), 449–474, Cambridge Univ. Press, New York, 2011]. The argument is framed as a discussion between a "Skeptic" and a "Set Theorist". The Skeptic's view is that uncountable sets lack genuine meaning, so set theory, and its extensions, do not reflect any mathematical reality, beyond the fact that its theorems can be understood as certain finitistic truths. The Set Theorist's position is that this view cannot explain the large cardinal hierarchy. Although well documented in the literature, not all of this position is stated explicitly in the paper; but part of it is that not only do large cardinal axioms fit together, but all extensions of set theory "considered in practice" seem to be equiconsistent with large cardinal axioms, their arithmetic consequences are compatible, and (eventually) so are their projective consequences, thus providing us, for example, with a coherent theory of the reals. As Woodin says, "the hierarchy of large cardinal axioms emerges as an intrinsic, fundamental conception within set theory". This is illustrated with the axiom of determinacy, an assertion about infinite games (pivotal to current research in set theory) that turns out to be equiconsistent with the assertion that there are infinitely many Woodin cardinals. The position elaborated in "The transfinite universe" is that strong extensions of set theory, such as ZF + AD, can only be justified through a dual process: first, the identification of a large cardinal assertion that is equiconsistent with the extension, and second, the understanding of the hierarchy of large cardinals as "true axioms about the universe of sets".

It is this last assertion that the Skeptic would attack. There are two objections here. One is that the unsolvable problems of set theory seem ubiquitous (e.g., is CH meaningful?). The other, the focus of the paper, is that all one is asserting about large cardinal axioms is formal consistency, and the axioms being considered are either consistent or "there is an elementary proof that [they] cannot hold". (Part of the point here is that there is no need to discuss "truth".) This is in a sense a serious challenge: Woodin presents an assertion, ZF + the existence of weak Reinhardt cardinals, that seems to subsume all known large cardinal axioms (for example, it implies the consistency of ZFC + there is a proper class of strongly $(\omega+1)$-huge cardinals). The cardinal κ is *weak Reinhardt* iff there are $\gamma > \lambda > \kappa$ such that

$$V_\kappa \prec V_\lambda \prec V_\gamma,$$

and there is an elementary embedding $j\colon V_{\lambda+2} \to V_{\lambda+2}$ with critical point κ. This contradicts choice, by the well-known Kunen inconsistency. The Set Theorist must therefore present a framework that would allow us to argue about the consistency of weak Reinhardt cardinals.

Woodin argues that such a framework is provided by the generic multiverse of sets, where rather than a fixed set theoretic universe, we consider the collection of all models reachable from a given "universe" (where large cardinals are present) by taking inner models (of ZF) and generic extensions. For more on the multiverse, see [J. D. Hamkins, Ann. Japan Assoc. Philos. Sci. **19** (2011), 37–55; MR2857736] and references therein. Woodin's version of the multiverse is discussed in more detail in "The continuum hypothesis, the generic-multiverse of sets, and the Ω conjecture" [W. H. Woodin, in *Set theory, arithmetic, and foundations of mathematics: theorems, philosophies*, 13–42, Lect. Notes Log., 36, Assoc. Symbol. Logic, La Jolla, CA, 2011; MR2882650]. It has the advantage of explaining CH and other "unsolvable" problems, and of giving us a way of dealing with models where choice fails (as symmetric extensions of models of choice). It also allows us to study all large cardinal assertions that can be expressed in a Σ_2 way (this is less restrictive than it first appears: for example, one can assert in a Σ_2 way that there is an α such that $V_\alpha \models$ there is a proper class of supercompact cardinals, even though the existence of one supercompact cardinal is not itself Σ_2).

The key observation is that the multiverse would have to refute the Ω conjecture if it is to validate the existence of weak Reinhardt cardinals. This is ultimately related to Tarski's undefinability of truth. For more on the Ω conjecture, see, for example, [J. Bagaria, N. Castells and P. B. Larson, in *Set theory*, 1–28, Trends Math., Birkhäuser, Basel, 2006; MR2267144] and references therein.

On the other hand, the Inner Model Program suggests an approach to validate the Ω conjecture. The idea of the program is to identify L-like models that accommodate large cardinals. The resulting rich structure can be seen in part as validating the adoption of the large cardinals under consideration. The standard approach is "from below", hence making it not a good candidate to discuss weak Reinhardt cardinals. Woodin's proposal of "ultimate L" approaches the program from above, by arguing that (modulo some technical conjectures) the development of a successful inner model theory at the level of supercompacts will actually provide us with the L-like model for all large cardinal assumptions. The technical details are discussed in "Suitable extender models I" [W. H. Woodin, J. Math. Log. **10** (2010), no. 1-2, 101–339; MR2802084] and "Suitable extender models II" [W. H. Woodin, J. Math. Log. **11** (2011), no. 2, 115–436, doi:10.1142/S021906131100102X].

Woodin proceeds to explain that, if successful, the identification of this "ultimate L" actually refutes the existence of weak Reinhardt cardinals. This is a very satisfactory response to the Skeptic's challenge: It restores the hierarchy of large cardinals, which can no longer be explained away as a collection of formal consequences of a single theory, and provides us with a robust universe from which a fruitful generic multiverse can be developed.

A word about the references is in order. All mentions of "Woodin (in press)" are to "The continuum hypothesis, the generic-multiverse of sets, and the Ω conjecture" [op. cit.]. Mentions of "Woodin (2009)" are as follows: On pages 90, 94, and 96, they point to "The transfinite universe" [op. cit.]. On pages 99, 101, and 116, they point to "Suitable extender models I, II" [op. cit.] (both papers were originally conceived as a unit). The mention on page 109 is a typo, and should be to [W. H. Woodin, in *One hundred years of Russell's paradox*, 29–47, de Gruyter Ser. Log. Appl., 6, de Gruyter, Berlin, 2004; MR2104736].

{For the collection containing this paper see MR2850464}

Andrés Eduardo Caicedo