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Proposing (ω_1, β)-morasses for $\omega_i \leq \beta$. (English summary)
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Morasses were first introduced by R. Jensen [see K. J. Devlin, *Constructibility*, Perspect. Math. Logic, Springer, Berlin, 1984; MR0750828; R. Jensen, "Higher-gap morasses", unpublished notes, 1972/73; per bibl.]. They describe an elaborate index set that allows us to obtain large direct limits from small structures. For example, structures of size $< \omega_\alpha$ indexed along an (ω_α, β)-morass give us a limit of size $\omega_{\alpha+\beta}$. This is used in model theory to prove cardinal transfer theorems. For example, Devlin's book explains how a gap-2 cardinal transfer theorem can be obtained from the existence of an $(\omega_1, 1)$-morass. Here, we say that a structure $\mathfrak{A} = (A, R^{\mathfrak{A}}, \ldots)$ is of type (κ, λ) iff $|A| = \kappa$ and $|R^{\mathfrak{A}}| = \lambda$, and a typical gap-$\tau$ theorem asserts that if $\mathfrak{A}$ is of type $(\kappa^{+\tau}, \kappa)$, then there is an elementary equivalent structure $\mathfrak{B}$ of type (ω_τ, ω) [see C. C. Chang and H. J. Keisler, *Model theory*, third edition, Stud. Logic Found. Math., 73, North-Holland, Amsterdam, 1990; MR1059055].

Jensen considered (κ, β)-morasses where κ is regular and $\beta < \kappa$; the paper under review provides references where the existence of morasses is established in L or in forcing extensions. The focus of the paper is to propose a generalization of the notion, where now $\beta \geq \kappa$. Specifically, the paper suggests a definition for (ω_1, β)-morasses. In Section 2, it is explained why this case was not considered previously. Briefly, an ordinal ν is called $\omega_{1+\beta}$*-like* iff $\{\alpha \mid L_\nu \models \alpha \text{ is an uncountable cardinal}\} \cup \{\nu\}$ has order type $\beta + 1$. The countable $\omega_{1+\beta}$-like ordinals are used as index sets in the definition of (ω_1, β)-morasses. Of course, there are no such countable ordinals if $\beta \geq \omega_1$.

Consider a family of structures $(\mathfrak{A}_\nu \mid \nu < \omega_1)$. Let $S_\alpha = \{\nu \mid L_\nu \models \alpha \text{ is the largest cardinal}\}$. If L_ν has a largest cardinal, call it α_ν. Jensen considered maps $f\colon S_{\alpha_{\overline{\nu}}} \cap \overline{\nu} \to S_{\alpha_\nu} \cap \nu$ under appropriate assumptions about $\overline{\nu}$ and ν. Let $M_{\overline{\nu}} = (\mathfrak{A}_\tau \mid \tau \in S_{\alpha_{\overline{\nu}}} \cap \overline{\nu})$. Under said assumptions, these maps can be extended to an embedding of $M_{\overline{\nu}}$ to M_ν, denoted by $f\colon \overline{\nu} \Rightarrow \nu$. The conditions require that if $\overline{\nu}$ is ω_γ-like, then so is ν ($\overline{\nu}$ and ν have the same type). In order to define his generalization, Irrgang allows that $\overline{\nu}$ and ν have different types, which requires a careful recursive construction, resembling the way that $\square$-sequences are defined in L, and indicates a close relationship between the existence of morasses and fine structure. The definition is involved, and is presented in detail in Section 2 (see particularly pp. 126–128).

To argue for the naturalness of the notion, Irrgang defines in Section 3 the related concept of κ*-standard morasses*, and shows that any $\omega_{1+\beta}$-standard morass is an (ω_1, β)-morass, and that the existence of a κ-standard morass implies that, for some sequence X, we have:

(1) There is an $S^X \subset \kappa$ consisting of limit ordinals and such that $X = (X_\nu \mid \nu \in S^X)$.
(2) For ν limit, let

$$I_\nu = \begin{cases} (J^X_\nu, X \restriction \nu) & \text{if } \nu \notin S^X, \\ (J^X_\nu, X \restriction \nu, X_\nu) & \text{if } \nu \in S^X, \end{cases}$$

and let $\beta(\nu)$ be the least β such that $J^X_{\beta+\omega} \models \nu$ is singular.

Then

(1) the structures I_ν are amenable;
(2) the X_ν are coherent: if $\nu \in S^X$, $H \prec_1 I_\nu$ and $\lambda = \sup(H \cap \mathrm{ORD})$, then $\lambda \in S^X$ and $X_\lambda = X_\nu \cap J^X_\lambda$;
(3) the I_ν satisfy condensation: if $\nu \in S^X$, and $H \prec_1 I_\nu$, then for some $\mu \in S^X$, $H \equiv I_\mu$;
(4) $S^X = \{\beta(\nu) \mid \nu \text{ is singular in } I_\kappa\}$; and
(5) the cardinals of $L_\kappa[X]$ coincide with the cardinals below κ.

In short, we say that $L_\kappa[X]$ computes cardinals correctly, and admits fine structure and condensation. As explained at the end of the paper, standard fine structural arguments can be carried out in this setting, which justifies the notation; in particular, there are Σ_n-Skolem functions for the I_ν.

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