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Caicedo, Andrés Eduardo (1-BOISE); **Friedman, Sy-David** (A-WIEN-IL)
BPFA and projective well-orderings of the reals. (English summary)
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Assuming the bounded proper forcing axiom (BPFA), the authors prove that if $\omega_1 = \omega_1^{L[r]}$ for some real r, then there is a $\Sigma^1_3(r)$ well-ordering of the reals. The proof combines a well-ordering due to Caicedo and Veličković with a coding method of David. The conclusion of this implication is best possible in the sense that, assuming MA_{ω_1}, there is no $\boldsymbol{\Sigma}^1_2$ well-ordering because $\boldsymbol{\Sigma}^1_2$ sets are Lebesgue measurable. It follows that the following two assertions are equiconsistent: (1) There is a proper class of reflecting cardinals, i.e., the cardinals κ for which V_κ is Σ_2-elementary in V. (2) Any forcing extension has a forcing extension where BPFA holds and there is a $\boldsymbol{\Sigma}^1_3$ well-ordering of the reals. Also, by results of Schindler it follows that if ω_1 is not a remarkable cardinal in L, then BPFA implies that there is a $\boldsymbol{\Sigma}^1_3$ well-ordering of the reals. Actually, the authors prove that, assuming MA_{ω_1} and $\omega_1 = \omega_1^{L[r]}$ for some real r, every Σ_1 relation on reals with ω_1 as a parameter is $\Sigma^1_3(r)$. *Miroslav Repický*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.