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★Regular cardinals without the weak partition property.
Wadge degrees and projective ordinals. The Cabal Seminar. Volume II, 509–517, *Lect. Notes Log.*, 37, *Assoc. Symbol. Logic, La Jolla, CA*, 2012.

The author works in the theory $\mathsf{ZF}+\mathsf{DC}$. In the reviewer's opinion, one of the most interesting *combinatorial* consequences of the axiom of determinacy is the existence of many cardinals with infinitary partition properties, though answering the question of their precise extent is still work in progress. The paper under review is concerned with this question.

For ordinals $\lambda \geq \vartheta$, denote by $(\lambda)^\vartheta$ the collection of (strictly) increasing maps $f\colon \vartheta \to \lambda$. For $\kappa \geq \lambda$, the partition relation

$$\kappa \to (\lambda)^\vartheta_\mu$$

holds iff for any partition $\mathcal{P}\colon (\kappa)^\vartheta \to \mu$ there is a homogeneous subset H of κ of type λ, where homogeneity means that $\mathcal{P}$ is constant on $(H)^\vartheta$. The subscript μ is omitted if $\mu = 2$. We write

$$\kappa \to (\lambda)^{<\vartheta}_\mu$$

iff $\kappa \to (\lambda)^\rho_\mu$ for all $\rho < \vartheta$. An ordinal κ satisfies the weak partition property iff $\kappa \to (\kappa)^{<\kappa}$. A stronger statement is that κ has the strong partition property (or: κ is a partition cardinal), which holds iff $\kappa \to (\kappa)^\kappa$. The negation of these properties is stated by replacing $\to$ with $\not\to$.

In work that goes back to Kleinberg and his collaborators [see E. M. Kleinberg, *Infinitary combinatorics and the axiom of determinateness*, Lecture Notes in Mathematics, Vol. 612, Springer, Berlin, 1977; MR0479903], it was shown that several consequences $(\star)$ of AD can be explained by showing that AD implies that certain cardinals have infinitary partition properties, and then that these partition properties suffice to imply $(\star)$. The connection between determinacy and partition properties runs deep [see for example A. S. Kechris and W. H. Woodin, in *Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I*, 355–378, Lect. Notes Log., 31, Assoc. Symbol. Logic, Chicago, IL, 2008; MR2463618].

It is well known that determinacy implies that many cardinals are measurable. Establishing infinitary partition properties is stronger. For example, if $\gamma \to (\gamma)^\omega_\lambda$ for all $\lambda < \gamma$, then γ is measurable, and this partition relation holds if γ has the weak partition property. Under determinacy, $\aleph_1$ is a partition cardinal, and $\aleph_2$ has the weak partition property. Work of Steel and Woodin shows that under determinacy, many regular cardinals below Θ are in fact measurable, and "many" can be replaced by "all" in (at least) all well-understood determinacy models [see J. R. Steel, Bull. Symbolic Logic **1** (1995), no. 1, 75–84; MR1324625]. In practice, proofs of regularity of cardinals below Θ under determinacy can be typically extended to proofs that the cardinals have infinitary partition properties.

In the paper under review, assuming determinacy, Jackson exhibits examples of regular cardinals without the weak partition property. Using his theory of *descriptions* of ordinals by measures, this gives us a detailed analysis of the partition properties satisfied by regular cardinals within the projective hierarchy. For the theory of descriptions, see [S. C. Jackson, in *Wadge degrees and projective ordinals. The Cabal Seminar. Volume II*, 199–269, Lect. Notes Log., 37, Assoc. Symbol. Logic, La Jolla, CA, 2012; MR2907000; and references therein].

To state the precise negative results in the paper requires some notation. First, given a measure μ, denote by j_μ the elementary embedding of HOD into the transitive collapse of its ultrapower by μ. (There is a small typo in the paper: The universe V is mentioned instead of HOD. The use of HOD is not essential here, and can be replaced by other structures as long as they can be well ordered, but recall that we are in a context where the full axiom of choice fails.)

Assume that κ is a partition cardinal and $\omega < \overline{\kappa} < \kappa$ is regular. The club filter on κ restricted to points of cofinality $\overline{\kappa}$ defines a normal measure ν on κ. Assume that, in addition, the club filter on $\overline{\kappa}$ restricted to points of cofinality ω defines a normal measure $\overline{\nu}$ of $\overline{\kappa}$. Suppose that κ is closed under $j_{\overline{\nu}}$, and let ϑ denote the ordinal corresponding to the class in the ultrapower of the function obtained by restricting $j_{\overline{\nu}}$ to κ, in symbols, $\vartheta = [j_{\overline{\nu}}|\kappa]_\nu$.

Theorem 2.1 shows that if $\lambda = j_\nu(\kappa)$, then $\lambda \not\to (\lambda)^\vartheta$. The argument is combinatorial in nature. It combines an analysis of the closure properties of the ultrapower of HOD by μ and ideas of the Martin-Paris theorem (see Corollary 13.3 of [A. S. Kechris, in *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, 91–132, Lecture Notes in Math., 689, Springer, Berlin, 1978; MR0526915]), together with a characterization of partition properties using club homogeneous sets. Though straightforward, the latter is key to Jackson's analysis of projective ordinals [see for example S. C. Jackson, in *Handbook of set theory. Vols. 1, 2, 3*, 1753–1876, Springer, Dordrecht, 2010; MR2768700].

(A notational warning: In the statements of Remark 2.3 and Lemma 2.4, $\mathcal{P}$ denotes the power set function, whereas, in their proofs, it denotes an appropriate partition.)

Determinacy is used to show that there are many pairs $\overline{\kappa} < \kappa$ satisfying the requirements above. For example, if $\kappa = \utilde{\delta}^1_3$, we can take $\overline{\kappa} = \omega_1$ (resulting in $\lambda = \aleph_{\omega\cdot 2+1}$) or $\overline{\kappa} = \omega_2$ (resulting in $\lambda = \aleph_{\omega^\omega+1}$). In general, the argument from Theorem 2.1 gives that under determinacy, if $\utilde{\delta}^1_{2n} < \lambda < \utilde{\delta}^1_{2n+1}$, then $\lambda \not\to (\lambda)^{\utilde{\delta}^1_{2n}}$.

Assume determinacy. Description theory is used to complement these results with some positive partition relations. In particular, Theorem 3.3 shows that for any regular λ in the interval $\utilde{\delta}^1_{2n} < \lambda < \utilde{\delta}^1_{2n+1}$ we have $\lambda \to (\lambda)^{<\utilde{\delta}^1_{2n}}$. The key is Lemma 3.1, which reproves a partial version of Theorem 7.1 of [A. S. Kechris and W. H. Woodin, in *Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I*, 379–397, Lect. Notes Log., 31, Assoc. Symbol. Logic, Chicago, IL, 2008; MR2463619], and also shows the following: Suppose that ν is the $\overline{\kappa}$-cofinal normal measure on $\utilde{\delta}^1_{2n-1}$, for some regular $\overline{\kappa} < \utilde{\delta}^1_{2n-1}$. Let $\lambda = j_\nu(\utilde{\delta}^1_{2n-1})$ be regular, $\utilde{\delta}^1_{2n-1} < \lambda < \utilde{\delta}^1_{2n+1}$. For all cardinals $\utilde{\delta}^1_{2n-1} \le \tau < \lambda$, the ultrapower of HOD by ν computes τ^+ correctly.

{For the collection containing this paper see MR2906066}

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