A review of sharps

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We hope this short note may prove useful as a guide to the general theory of sharps. Only a knowledge of the theory of $0^\sharp$ is required. This note will be updated periodically, the original version was part of the introduction to the author's dissertation [1], written under the supervision of John Steel and Hugh Woodin at U.C. Berkeley.

Definition 0.1. Let Y be a transitive set.

1. A *class of indiscernibles* for $L(Y), Y$ (informally, for $L(Y)$) is a class $I \subseteq$ ORD such that for all $\vec{a}$ elements of Y and all $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ elements of I, for any $\varphi(\vec{x}, y_1, \dots, y_n)$ in the language of set theory,
$$L(Y) \models \varphi(\vec{a}, \vec{\alpha}) \iff L(Y) \models \varphi(\vec{a}, \vec{\beta}).[1]$$
2. Let $\varphi(t, x_1, \dots, x_n)$ be a formula in the language of set theory, expanded with constant symbols for Y and the elements of Y. A *weak Skolem function* for φ (with respect to $L(Y), Y$) is the function $f_\varphi : {}^nL(Y) \to L(Y)$ given by
$$f_\varphi(\vec{x}) = \begin{cases} y & \text{if } L(Y) \models y \texttt{ is the unique } z \texttt{ such that } \varphi(z, \vec{x}); \\ \emptyset & \text{if there is no such unique } z. \end{cases}$$
3. Let $Y \subseteq Z \subseteq L(Y)$. By $\mathcal{H}(L(Y), Z)$ we mean the closure of Z under weak Skolem functions.
4. Let I be a class of indiscernibles for $L(Y), Y$. We say that I *generates* $L(Y)$ iff
$$\mathcal{H}(L(Y), I \cup Y) = L(Y).$$
5. We say that $Y^\sharp$ *exists* iff there is a club proper class I of indiscernibles for $L(Y), Y$ such that $I \cup Y$ generates $L(Y)$ and, moreover, for any uncountable η such that $Y \in H_\eta$, $\mathcal{H}(L(Y), (I \cap \eta) \cup Y) = L_\eta(Y)$.
6. We say that $X^\sharp$ exists iff $Y^\sharp$ exists, where $Y = \text{Tr. Cl.}(X)$.

Fact 0.2. *If $X \in H_\eta$ and η is Ramsey, then $X^\sharp$ exists.* $\Box$

[1] We consider the language of $L(Y)$ to be expanded by constants P_a for each $a \in Y \cup \{Y\}$. The natural interpretation of P_a is, of course, the set a.

The assertion "$X^\sharp$ exists" refers to the existence of a proper class object. Solovay's realization (see [2]) is that just as in the case of sharps for reals, this is in fact equivalent to the existence of a set, and it is this set what we now call $X^\sharp$.

Definition 0.3. Let Y be transitive.

1. Let $\mathcal{L}_Y$ denote the language of set theory augmented with constants for the elements of $Y \cup \{Y\}$, and with ω many other constants c_n, $n \in \omega$, (to represent the first ω indiscernibles), and closed under terms for weak Skolem functions for formulas in the language of set theory.

2. An *EM blueprint* for Y (EM stands for Ehrenfeucht-Mostowski) is the theory in $\mathcal{L}_Y$ of some structure $(L_\eta(Y), \in, P_a, i_n : a \in Y \cup \{Y\}, n \in \omega)$ where $Y \in H_\eta$ or $\eta = \text{ORD}$, and $\langle\, i_n : n < \omega \,\rangle$ is the increasing enumeration of a set of indiscernibles for
$$(L_\eta(Y), \in, P_a)_{a \in Y \cup \{Y\}}.$$

3. Let Σ be an EM blueprint for Y, and let α be an ordinal. By $\Gamma(\Sigma, \alpha)$ we mean, provided that it exists and is unique (up to isomorphism), a model $\mathcal{M}_\alpha$ such that
   (a) $\mathcal{M}_\alpha \models \Sigma^*$, the restriction of Σ to the language $\mathcal{L}'_Y$ without constants for the indiscernibles.
   (b) There is a set $I \subset \text{ORD}^{\mathcal{M}_\alpha}$ such that $(I, \in^{\mathcal{M}_\alpha}) \cong (\alpha, \in)$ which is a set of indiscernibles for $\mathcal{M}_\alpha$.
   (c) $\mathcal{H}(\mathcal{M}_\alpha, I \cup \{\, P_a^{\mathcal{M}_\alpha} : a \in Y \cup \{Y\} \,\}) = \mathcal{M}_\alpha$.

4. A set of sentences $\Sigma \subseteq \mathcal{L}_Y$ is a *remarkable character* for Y iff
   (a) Σ is an EM blueprint for Y. In fact, Σ extends "$\mathsf{ZF} + V = L(Y)$".
   (b) $\Gamma(\Sigma, \alpha)$ exists and is well-founded for all α.
   (c) For any term $t(x_0, \ldots, x_{n-1})$ in $\mathcal{L}_Y$, the sentence
   $$\text{“}t(c_0, \ldots, c_{n-1}) \in \text{ORD} \longrightarrow t(c_0, \ldots, c_{n-1}) < c_n\text{”}$$
   belongs to Σ.
   (d) For any term $t(x_0, \ldots, x_{m+n})$ in $\mathcal{L}_Y$, the sentence
   $$\text{“}t(c_0, \ldots, c_{m+n}) < c_m \longrightarrow t(c_0, \ldots, c_{m+n}) = t(c_0, \ldots, c_{m-1}, c_{m+n+1}, \ldots, c_{m+2n+1})\text{”}$$
   belongs to Σ.
   (e) Σ satisfies the *witness condition*:

   Whenever $\exists x\, \varphi(x) \in \Sigma$, there is a term t all of whose constants for indiscernibles already appear on $\varphi(x)$, and such that $\varphi(t) \in \Sigma$.

The witness condition is the key condition that remarkable characters for reals (or more generally for sets of ordinals) satisfy automatically, because Skolem terms are definable in $L[x]$, $x \in \mathbb{R}$, since $L[x]$ has a definable well-ordering. Its importance lies in that it allows us to prove the following basic fact:

Lemma 0.4 (Solovay). *If Σ is a remarkable character for a transitive set Y, then*

1. *For all α, the sequence I^α of indiscernibles of $\Gamma(\Sigma,\alpha)$ with*
$$(I^\alpha, \in^{\Gamma(\Sigma,\alpha)}) \cong (\alpha, \in)$$
satisfies that for any formula $\varphi(x_1,\dots,x_n)$ in the language $\mathcal{L}'_Y$,
$$\varphi(c_1,\dots,c_n) \in \Sigma$$
iff there is a $\in^{\Gamma(\Sigma,\alpha)}$-increasing sequence $a_1,\dots,a_n$ of elements of I^α such that $\Gamma(\Sigma,\alpha) \models \varphi(a_1,\dots,a_n)$.
2. *For any cardinal η such that $Y \in H_\eta$,*
$$\Gamma(\Sigma,\eta) \cong L_\eta(Y).$$
3. *For all α, I^α is closed unbounded in $\mathrm{ORD}^{\Gamma(\Sigma,\alpha)}$.*
4. *If $\alpha < \beta$, then I^β end-extends I^α (seen as subsets of $\mathrm{ORD}^{L_\eta(Y)}$ for any cardinal η such that $\beta, Y \in H_\eta$.)*
5. *For any η such that $Y \in H_\eta$,*
$$\mathcal{H}(L(Y), I^\eta \cup Y) = L_\eta(Y) \prec \mathcal{H}(L(Y), \bigcup_\alpha I^\alpha \cup Y) = L(Y).$$
6. *Let Σ' be any remarkable character for Y. Then $\Sigma' = \Sigma$.* $\square$

Corollary 0.5 (Solovay). *Let Y be transitive. Then $Y^\sharp$ exists iff there is a remarkable character for Y.* $\square$

Remark 0.6. In truth, Solovay only argued these results for sharps of sets of reals (or, more precisely, for $\mathbb{R}^\sharp$), but the arguments for $0^\sharp$ lift straightforwardly.

It follows that it makes sense to define sharps in terms of the remarkable characters whose existence they guarantee:

Definition 0.7. Let X be a set and let Y be its transitive closure. Then $X^\sharp := \Sigma$, for Σ the unique remarkable character for Y.

See [2], where the general notion of sharps is introduced, in the context of subsets of reals.

Notice the definition of $Y^\sharp$ is absolute in the sense that if $W \supseteq V$ is an outer model and $Y^\sharp \in V$, then
$$W \models (Y^\sharp)^V = Y^\sharp.$$

The following is ancient, but I have been unable to find a reference:

Fact 0.8. *Let $\mathbb{P}$ be a poset, and suppose $x^\sharp \in V^{\mathbb{P}}$, where x is a real coding a set $X \in V$. Then $X^\sharp \in V$.* $\square$

It follows from the fact that Jensen's covering lemma relativizes to all sharps, so $L[X]$ satisfies covering above η, where $X \in H_\eta$, iff $X^\sharp$ does not exist. Since set sized forcing preserves a tail of the class of cardinals, if $\mathbb{P}$ is a poset and $X^\sharp$ exists in $V^{\mathbb{P}}$, then $X^\sharp$ exists in V.

Fact 0.9 (Solovay). *If $X^\sharp$ exists, then the truth sets of $L(X)$ and $L[X]$ are definable.* $\square$

The following example must be folklore, it was shown to me by Woodin. It illustrates that we cannot make do in the definition of $X^\sharp$ without the witness condition:

Recall first that after adding ω_1 Cohen reals, no well-ordering of $\mathbb{R}$ belongs to $L(\mathbb{R})$. This follows immediately from the weak homogeneity of the forcing, call it $\mathbb{P}$, and the fact that $\mathbb{P}$ is ccc and $\mathbb{P} \cong \mathbb{P} \times \mathbb{P}$. From this, an elementary argument shows that, in fact, there is in $V^{\mathbb{P}}$ a set $\mathbb{R}_1 \subsetneq \mathbb{R}^{V^{\mathbb{P}}}$ and an elementary embedding $j : L(\mathbb{R}_1) \to L(\mathbb{R}^{V^{\mathbb{P}}})$ that fixes the ordinals, so in particular Choice fails in $L(\mathbb{R}^{V^{\mathbb{P}}})$ and the result follows.

Claim 0.10. *Let $V = L[\mu]$ be the smallest inner model for a measurable cardinal and let G be $\mathtt{Add}(\omega, \omega_1)$-generic over V. Then*

1. *$(\mathbb{R}^\sharp)^{V[G]}$ exists.*
2. *$(\mathbb{R}^\sharp)^{V[G]} \cap V \in V$.*
3. *$(\mathbb{R}^\sharp)^{V[G]} \cap V$ satisfies conditions 4.(a)–(d) of Definition 0.3 for $(\mathbb{R}^\sharp)^V$.* $\square$

If we could dispense with the witness condition in Definition 0.3, it would follow from the claim that $\mathbb{R}^{L[\mu]}$ is not well-orderable by a well-ordering in $L(\mathbb{R})^{L[\mu]}$. This is absurd, since in fact $\mathbb{R}^{L[\mu]}$ admits a Δ^1_3-well-ordering.

Remark 0.11. Of course, the same arguments generalize to larger sharp-like objects, like daggers or pistols.

The theory of sharps is usually recalled in connection with finestructural arguments. In this context, $X^\sharp$ is usually defined as a particular kind of mouse.

Fact 0.12. *Let X be a set. Then $X^\sharp$ exists iff there is an active X-mouse.* $\square$

There is therefore no lack of generality in using this approach. We actually obtain quite more information than what was stated in Fact 0.12. For example, by standard techniques a mouse as in 0.12 is unique if it exists, and so we can identify it with $X^\sharp$. Moreover, for example if $x \in \mathbb{R}$, $x^\sharp$ and the minimal active x-mouse share the same Turing degree.

References

[1] A. Caicedo. *Simply definable well-orderings of the reals*, Ph. D. Dissertation, Department of Mathematics, University of California, Berkeley (2003).

[2] R. Solovay. *The independence of* DC *from* AD, in **Cabal Seminar 76–77: Proceedings, Caltech-UCLA logic seminar 1976–77**, A. Kechris and Y. Moschovakis, eds., Springer-Verlag (1978), 171–183.