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Real-valued measurable cardinals and well-orderings of the reals. (English) [\[Zbl 1111.03044\]](#)
Bagaria, Joan (ed.) et al., Set theory. Centre de Recerca Matemàtica Barcelona, 2003–2004. Basel: Birkhäuser (ISBN 3-7643-7691-0/hbk). Trends in Mathematics, 83-120 (2006).

Summary: We show that the existence of atomlessly measurable cardinals is incompatible with the existence of well-orderings of the reals in $L()$, but consistent with the existence of well-orderings of the reals that are third-order definable in the language of arithmetic. Specifically, we provide a general argument that, starting from a measurable cardinal, produces a forcing extension where \mathfrak{c} is real-valued measurable and there is a Δ_2^2 -well-ordering of \mathbb{R} . A variation of this idea, due to Woodin, gives Σ_1^2 -well-orderings when applied to $L[\mu]$ or, more generally, $\Sigma_1^2(\text{Hom}_\infty)$ if applied to nice inner models, provided enough large cardinals exist in V . We announce a recent result of Woodin indicating how to transform this variation into a proof from large cardinals of the Ω -consistency of real-valued measurability of \mathfrak{c} together with the existence of Σ_1^2 -definable well-orderings of \mathbb{R} . It follows that if the Ω -conjecture is true, and large cardinals are granted, then this statement can always be forced.

However, we introduce a strengthening of real-valued measurability (real-valued hugeness), show its consistency, and prove that it contradicts the existence of third-order definable well-orderings of \mathbb{R} .

For the entire collection see [\[Zbl 1097.03004\]](#).

MSC:

- [03E35](#) Consistency; independence results (set theory)
- [03E45](#) Constructibility, ordinal definability, and related notions
- [03E55](#) Large cardinals