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A trichotomy theorem in natural models of AD^+. (English) Zbl 1245.03079

Babinkostova, L. (ed.) et al., Set theory and its applications. Annual Boise extravaganza in set theory, Boise, ID, USA, 1995–2010. Providence, RI: American Mathematical Society (AMS) (ISBN 978-0-8218-4812-8/pbk). Contemporary Mathematics 533, 227-258 (2011).

Summary: Assume AD^+ and that either $V = L(\mathcal{P}())$, or $V = L(T,)$ for some set $T \subset \mathrm{ORD}$. Let $(X, \leq)$ be a pre-partially ordered set. Then exactly one of the following cases holds: (1) X can be written as a well-ordered union of pre-chains, or (2) X admits a perfect set of pairwise $\leq$-incomparable elements, and the quotient partial order induced by $(X, \leq)$ embeds into $(2^\alpha, \leq_{\mathrm{lex}})$ for some ordinal α, or (3) there is an embedding of $2^\omega/E_0$ into $(X, \leq)$ whose range consists of pairwise $\leq$-incomparable elements.

By considering the case where $\leq$ is the diagonal on X, it follows that for any set X exactly one of the following cases holds: (1) X is well-orderable, or (2) X embeds the reals and is linearly orderable, or (3) $2^\omega/E_0$ embeds into X. In particular, a set is linearly orderable if and only if it embeds into $\mathcal{P}(\alpha)$ for some α. Also, ω is the smallest infinite cardinal, and $\{\omega_1, \}$ is a basis for the uncountable cardinals.

Assuming the model has the form $L(T,)$ for some $T \subset \mathrm{ORD}$, the result is a consequence of $\mathrm{ZF} + \mathrm{DC}$ together with the existence of a fine σ-complete measure on $\mathcal{P}_{\omega_1}()$ via an analysis of Vopěnka-like forcing. It is known that in the models not covered by this case, AD holds. The result then requires more of the theory of determinacy; in particular, that $V = \mathrm{OD}((< \Theta)^\omega)$, and the existence and uniqueness of supercompactness measures on $\mathcal{P}_{\omega_1}(\gamma)$ for $\gamma < \Theta$.

As an application, we show that (under the same basic assumptions) Scheepers's countable-finite game over a set S is undetermined whenever S is uncountable.

For the entire collection see [Zbl 1205.03004].

MSC:

03E60 Axiom of determinacy, etc.
03E25 Axiom of choice and related propositions (logic)
03C20 Ultraproducts and related constructions