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Real-valued Measurable Cardinals and Well-orderings of the Reals

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Abstract. We show that the existence of atomlessly measurable cardinals is incompatible with the existence of well-orderings of the reals in $L(\mathbb{R})$, but consistent with the existence of well-orderings of the reals that are third-order definable in the language of arithmetic. Specifically, we provide a general argument that, starting from a measurable cardinal, produces a forcing extension where $\mathfrak{c}$ is real-valued measurable and there is a Δ^2_2-well-ordering of $\mathbb{R}$. A variation of this idea, due to Woodin, gives Σ^2_1-well-orderings when applied to $L[\mu]$ or, more generally, $\Sigma^2_1(\mathrm{Hom}_\infty)$ if applied to nice inner models, provided enough large cardinals exist in V. We announce a recent result of Woodin indicating how to transform this variation into a proof from large cardinals of the Ω-consistency of real-valued measurability of $\mathfrak{c}$ together with the existence of Σ^2_1-definable well-orderings of $\mathbb{R}$. It follows that if the Ω-conjecture is true, and large cardinals are granted, then this statement can always be forced.

However, we introduce a strengthening of real-valued measurability (real-valued hugeness), show its consistency, and prove that it contradicts the existence of third-order definable well-orderings of $\mathbb{R}$.

This work deals with consistency results within the theory of real-valued measurable cardinals and draws from Chapter 3 of the author's dissertation [11], written at the University of California, Berkeley, under the supervision of John R. Steel and W. Hugh Woodin. The author wishes to thank both of them for their guidance and patience. He also wishes to thank the referee for comments that helped to improve the presentation significantly.

1. Basics of random forcing

This section is included in order to make this paper reasonably self-contained, and we do not claim much originality here other than by way of exposition. The main references for the theory of real-valued measurable cardinals are [38] and [19], see also [32] and [22]. For whatever modest contributions in this section are due to us, see after Fact 1.27. Our notation is mostly standard, see [24], [31], and [26] for whatever notions we leave undefined. ZFC^- denotes ZFC without the Power-Set axiom. We start by defining our basic objects:

Definition 1.1. A cardinal κ is *real-valued measurable*, $\mathsf{RVM}(\kappa)$, iff it is uncountable and there is a κ-additive probability measure $\nu : \mathcal{P}(\kappa) \to [0,1]$ that is null on singletons. We call ν a *witnessing* probability.

A real-valued measurable cardinal κ is *atomlessly measurable* iff there is an atomless witnessing probability ν.

That ν is κ-additive means that whenever $\gamma < \kappa$ and $\langle A_\alpha : \alpha < \gamma \rangle$ is a sequence of disjoint subsets of κ, then

$$\nu\left(\bigcup_{\alpha<\gamma} A_\alpha\right) = \sum_{\alpha<\gamma} \nu(A_\alpha) := \sup\left\{\sum_{\alpha\in\mathcal{F}} \nu(A_\alpha) : \mathcal{F} \subset \gamma \text{ is finite}\right\}.$$

Of course, this implies in particular that only countably many of the A_α have positive measure: Otherwise, for some n,

$$B_n = \left\{\alpha < \gamma : \nu(A_\alpha) > \frac{1}{n+1}\right\}$$

would be infinite, contradicting that ν is bounded above by 1. See also Claim 1.30.

That ν is atomless means that whenever $0 < \nu(A)$, there is $B \subset A$ with $0 < \nu(B) < \nu(A)$. We leave it as an easy exercise for the reader to see that in this case, for any ε with $0 < \varepsilon < \nu(A)$, there is $B \subset A$ with $\nu(B) = \varepsilon$ (or see [26, Lemma 2.6] for a hint on how to proceed).

The following is due to Ulam [43], who also introduced the concept:

Theorem 1.2. *If* $\mathsf{RVM}(\kappa)$, *then* κ *is either measurable or atomlessly measurable, in which case* $\kappa \leq \mathfrak{c}$. □

Definition 1.3. Let ν be a complete measure on some set X. Then

$$\mathcal{N}_\nu := \{\, Y \subseteq X : \nu(Y) = 0 \,\}$$

is the ideal of *ν-null sets*.

Since $\mathrm{add}(\mathcal{N}_\nu)$ is necessarily a regular cardinal, we have the following useful fact:

Fact 1.4. *Suppose* $\mathsf{RVM}(\kappa)$ *and* ν *is a witnessing probability. Then:*

1. $\kappa = \mathrm{add}(\mathcal{N}_\nu)$ *is regular.*
2. $\mathcal{N}_\nu$ *is an* $\aleph_1$*-saturated ideal on* κ. □

Remark 1.5. In fact, if $\kappa \leq \mathfrak{c}$ is real-valued measurable, then κ is weakly Mahlo, the κ^{th} weakly Mahlo, etc. Recall that κ is *weakly Mahlo* iff it is uncountable and

$$\{\, \rho < \kappa : \rho \text{ is regular} \,\}$$

is stationary in κ. One can see this as a corollary of Theorem 1.6, see Corollary 1.24. That κ is weakly inaccessible follows immediately from Fact 1.4 and the existence of Ulam matrices on successor cardinals, see [31, Theorem II.6.11].

The following basic characterization is due to Solovay, and will be essential for our arguments:

Theorem 1.6. $\mathsf{RVM}(\kappa)$ *iff there is* $\lambda \geq \omega_1$ *such that*

$$V^{\mathtt{Random}_\lambda} \models \exists j : V \overset{\prec}{\longrightarrow} N, \quad \mathrm{cp}(j) = \kappa,$$

where $\mathtt{Random}_\lambda$ *is the forcing for adding* λ *many random reals.*

If $\kappa \leq \mathfrak{c}$ *and* $\mathsf{RVM}(\kappa)$*, then in addition we can require that*

$$V^{\mathtt{Random}_\lambda} \models {}^{\omega}N \subseteq N.$$

As far as the author can see, the statement of Theorem 1.6 has not appeared explicitly in print. It can certainly be glimpsed in the arguments of [38] (see especially [38, §6]) and it is well known to experts in the area, see for example [22].

Definition 1.7. Specifically, $\mathtt{Random}_\lambda$ is the collection of Borel subsets of 2^λ, modulo null sets, where the measure φ is defined as follows:

- For $J \subset \lambda$, J finite, and $z \in 2^J$, the *cylinder* determined by J, z is
$$C = C_{J,z} := \{\, x \in 2^\lambda : x{\restriction}J = z \,\}.$$
For such a C, define $\varphi(C) := 2^{-|J|}$.
- The cylinders generate the product topology on 2^λ. Extend φ to a Borel measure by:
$$\varphi(B) := \inf\left\{ \sum_n \varphi(C_n) : B \subseteq \bigcup_n C_n,\ C_n \text{ a cylinder} \right\}$$
for B a Borel subset of 2^λ.

Remark 1.8. In fact, we can extend φ to a complete measure in the standard way. Some presentations of random forcing assume that we are working with this completion and not just with its restriction to Borel sets. For the purposes of forcing, the resulting Boolean algebras are equivalent, and we can ignore the difference.

Definition 1.9. Let $\mathbb{B}$ be a σ-complete Boolean algebra. A *'probability measure'* on $\mathbb{B}$ is a function $\nu : \mathbb{B} \to [0,1]$ such that

1. $\nu(a) = 0$ iff $a = \mathsf{0}$.
2. $\nu(\mathsf{1}) = 1$.
3. ν is σ-additive: If $\{\, a_n : n \in \omega \,\}$ is an antichain in $\mathbb{B}$, so $a_n \cdot a_m = \mathsf{0}$ whenever $n \neq m$, then
$$\nu\left(\sum_n^{\mathbb{B}} a_n \right) = \sum_n \nu(a_n).$$

We call $(\mathbb{B}, \nu)$ a *measure algebra.*

Fact 1.10.

1. *For all* λ, $\mathtt{Random}_\lambda$ *is ccc and, therefore, a complete Boolean algebra.*

2. *The map* $\nu : \mathtt{Random}_\lambda \to [0,1]$ *given by* $\nu([X]) = \varphi(X)$*, where* φ *is as described above and* $[X]$ *denotes the equivalence class of the Borel subset* $X \subseteq 2^\lambda$*, is a 'probability measure', so* $(\mathtt{Random}_\lambda, \nu)$ *is a measure algebra.*

Proof. That $\mathtt{Random}_\lambda$ is ccc follows from Claim 1.30. Since it is σ-complete and ccc, it is a complete Boolean algebra. A proof of 2 can be found in [18], see Remark 1.11 below. □

Remark 1.11. Given any probability space $(X, \mathcal{P}, \mu)$, $\mathcal{P}/\mathcal{N}_\mu$ can be turned into a measure algebra by exactly the same construction as in *2* of Fact 1.10, see [18]. More significantly,

Fact 1.12. *Any measure algebra is isomorphic (as measure algebra) to one of the form* $\mathcal{P}/\mathcal{N}_\mu$ *for some probability space* $(X, \mathcal{P}, \mu)$*, where* $\mathcal{P}/\mathcal{N}_\mu$ *is a measure algebra with the 'probability measure' described in Fact* 1.10.2. □

This is a consequence of the so-called Loomis-Sikorski theorem (due to von Neumann) stating that any σ-complete Boolean algebra is isomorphic (as a Boolean algebra) to $\Sigma/\mathcal{I}$ for some σ-algebra Σ of subsets of some set X, and some σ-complete ideal $\mathcal{I}$ on Σ. See [27] and [18] for details.

Definition 1.13.

1. For $\mathbb{B}$ a complete Boolean algebra, the *generating number* of $\mathbb{B}$ is $\tau(\mathbb{B}) := \min\{\, |X| : X \text{ generates } \mathbb{B} \text{ (as a complete algebra)} \,\}$.
2. $\mathbb{B}$ is τ*-homogeneous* iff[1] $\tau(\mathbb{B}) = \tau(\mathbb{B}{\restriction}a)$ for any $a \neq \mathtt{0}$.

Fact 1.14.

1. *If* $\mathbb{B}$ *is a complete Boolean algebra which is homogeneous in the forcing sense*[2]*, then* $\mathbb{B}$ *is* τ*-homogeneous.*
2. *Let* λ *be a cardinal. Then* $\mathtt{Random}_\lambda$ *is homogeneous. Thus, it is* τ*-homogeneous, and* $\tau(\mathtt{Random}_\lambda) = \lambda$. □

Theorem 1.15 (Maharam, see [18, Theorem 3.8]**).** *If* $\mathbb{B}$ *is a complete* τ*-homogeneous measure algebra, then it is isomorphic as a measure algebra to exactly one* $\mathtt{Random}_\lambda$ *up to the cardinality of* λ. □

Maharam's theorem is actually much more general than we have stated, but this particular case is all we need.

Fact 1.16. *If* $\mathbb{B} \lessdot \mathtt{Random}_\lambda$ *(i.e.,* $\mathbb{B}$ *is a complete subalgebra of* $\mathtt{Random}_\lambda$*), then there is a condition* $p \in \mathbb{B}$ *(equivalently, there is a dense set of such conditions) such that* $\mathbb{B}{\restriction}p \cong \mathtt{Random}_\gamma$ *for some* γ. □

Notice that, conversely, if $\gamma < \lambda$, then $\mathtt{Random}_\gamma \lessdot \mathtt{Random}_\lambda$.

[1] For $p \in \mathbb{B} \setminus \{\mathtt{0}\}$, $\mathbb{B}{\restriction}p$ is the Boolean algebra of elements of $\mathbb{B}$ below p.

[2] I.e., for any $p, q \in \mathbb{B} \setminus \{\mathtt{0}\}$ there are $\mathtt{0} < r \leq p$ and $\mathtt{0} < s \leq q$ such that $\mathbb{B}{\restriction}r$ and $\mathbb{B}{\restriction}s$ are isomorphic.

Remark 1.17. The version of Fact 1.16 for Cohen forcing is true for $\lambda \leq \omega_1$ (see [28] or [6]) but false for $\lambda \geq \omega_2$, see [29].

The following is [32, Theorem 3.13].

Fact 1.18. *Let* $\mathbb{B} \lessdot \mathtt{Random}_\lambda$*. Then*

$$\mathsf{1} \Vdash_{\mathbb{B}} (\mathtt{Random}_\lambda)^V/\mathbb{B} \cong \dot{\mathtt{Random}}_\gamma \text{ for some } \gamma. \qquad \square$$

The following is [32, Lemma 3.12].

Fact 1.19. *If* $W \supseteq V$ *is an outer model and* G *(identified as a subset of* λ*) is* $(\mathtt{Random}_\lambda)^W$*-generic over* W*, then* G *is* $(\mathtt{Random}_\lambda)^V$*-generic over* V*. In particular, for any* $\mathbb{P}$*,* $\mathtt{Random}_\lambda$ *completely embeds into* $\mathbb{P} * \dot{\mathbb{Q}}$*, where* $\dot{\mathbb{Q}}$ *is a* $\mathbb{P}$*-name for* $(\mathtt{Random}_\lambda)^{V^{\mathbb{P}}}$*.* $\square$

Proof of Solovay's Theorem 1.6. $(\Leftarrow)$ Suppose

$$V^{\mathtt{Random}_\lambda} \models \exists j : V \xrightarrow{\prec} N, \quad \mathrm{cp}(j) = \kappa.$$

Let $\varphi : \mathtt{Random}_\lambda \to [0,1]$ be the 'probability measure' associated to $\mathtt{Random}_\lambda$. In V, we want to define a probability measure on subsets of κ. Fix names j and N such that

$$[\![N \text{ is a transitive inner model and } j : V \xrightarrow{\prec} N, \mathrm{cp}(j) = \kappa]\!] = \mathsf{1}.$$

For $A \subseteq \kappa$, let $\nu(A) := \varphi[\![\kappa \in j(A)]\!]$, so $\nu : \mathcal{P}(\kappa) \to [0,1]$. It is easy to verify that ν is as wanted[3].

$(\Rightarrow)$ Suppose $\mathsf{RVM}(\kappa)$. Let ν be a witness, and let $\mathbb{B}_\nu = \mathcal{P}(\kappa)/\mathcal{N}_\nu$. Since $\mathcal{N}_\nu$ is $\aleph_1$-saturated, $\mathbb{B}_\nu$ is complete (by the Smith-Tarski theorem [26, Proposition 16.5]), and we may assume (by reducing to a subset if necessary) that $\mathbb{B}_\nu \cong \mathtt{Random}_\lambda$ for some λ: Necessarily, for some $X \subseteq \kappa$, $X \notin \mathcal{N}_\nu$, we must have that $\mathcal{P}(X)/\mathcal{N}_\nu$ is τ-homogeneous because τ is a decreasing ordinal-valued function and therefore eventually constant. By replacing ν with $\hat{\nu} : Y \mapsto \nu(X \cap Y)$, we may as well assume $X = \kappa$. That $\mathbb{B}_\nu \cong \mathtt{Random}_\lambda$ for some λ now follows from Maharam's Theorem 1.15. If $\kappa \leq \mathfrak{c}$ then $|\mathbb{B}_\nu| \geq \mathfrak{c}$.

Let G be $\mathbb{B}_\nu$-generic over V. Then G is essentially a V-ultrafilter on κ, and we can form $\pi : V \to \mathrm{Ult}(V, G)$ in $V[G]$. But the saturation of $\mathcal{N}_\nu$ ensures that the ultrapower is well founded, and therefore isomorphic to a transitive class N. Let $j : V \xrightarrow{\prec} N$ denote the corresponding embedding, coming from π via the Mostowski collapse. Then $\mathrm{cp}(j) = \kappa$, and since $\mathbb{B}_\nu \cong \mathtt{Random}_\lambda$, we are done, except for the claim that $\lambda \geq \omega_1$. For this, see [21, §2], where it is shown that in fact $\lambda \geq \kappa^+$.

See Fact 1.20 and Remark 1.21 for the proof that ${}^\omega N \subseteq N$. $\square$

Fact 1.20. *Suppose* $\mathsf{RVM}(\kappa)$ *and* $\mathtt{Random}_\lambda$*,* j *and* N *are as in Solovay's theorem. Then* $\mathbb{R}^N = \mathbb{R}^{V^{\mathtt{Random}_\lambda}}$*.*

[3] Those uncomfortable with our use of proper classes are advised to consult [38] for a first-order treatment.

Proof. This is standard from the theory of saturated ideals: In fact, using the notation from the theorem, if G is $\mathbb{B}_\nu$-generic over V, then $V[G] \models {}^\omega N \subseteq N$. $\square$

Remark 1.21. A strong version of Fact 1.20 is that we can in fact assume that ${}^\kappa N \subset N$:

Use notation as above. We claim first that for every term $\dot{b}$ in $\mathtt{Random}_\lambda$ for an element of the ground model V, there is a function $f \in V$ such that

$$[\![[f]_N = j(\dot{b})]\!].$$

In effect, suppose $[\![\dot{b} \in V]\!] = \mathtt{1}$ and let $A = \{ a_\xi : \xi \text{ is a possible value of } \dot{b} \} \in V$ be a maximal antichain in $\mathcal{P}(\kappa)/\mathcal{N}_\nu$, so for each $a_\xi \in A$, $a_\xi \Vdash \dot{b} = \xi$. Since $\mathcal{N}_\nu$ is ω_1-saturated, A is countable, so we may assume A is a partition of κ, i.e., $a_\xi \subseteq \kappa$ for each $a_\xi \in A$ and $a_\xi \cap a_\zeta = \emptyset$ whenever $\xi \neq \zeta$. In V, define $f : \kappa \to V$ by

$$f(\eta) = \text{the unique } \xi \text{ such that } \eta \in a_\xi.$$

Then $[\![[f]_N = j(\dot{b})]\!] = \mathtt{1}$, since for any ξ, $a_\xi \Vdash [f]_N = [c_\xi]_N$, so $\mathtt{1} \Vdash [f]_N = [c_{\dot{b}}]_N = j(\dot{b})$. This easily leads to a proof that, in $\mathtt{Random}_\lambda$, N is closed under ω-sequences and, in fact, under sequences of length $< \kappa$.

Without loss of generality, the null ideal $\mathcal{N}_\nu$ is normal (see Corollary 1.23), so the identity represents κ in the ultrapower N. Assuming normality of $\mathcal{N}_\nu$, we prove that it is in fact closed under κ-sequences.

Given any term $\langle \tau_\alpha : \alpha < \kappa \rangle$ for a κ-sequence in $V[G]$ of elements of N, let $\langle \rho_\alpha : \alpha < \kappa \rangle$ be a term for a κ-sequence of functions in V such that for each α,

$$[\![\rho_\alpha \in V \cap {}^\kappa V \text{ and } \tau_\alpha = [\rho_\alpha]_N]\!] = \mathtt{1}$$

and then a sequence $\langle f_\alpha : \alpha < \kappa \rangle$ of functions $f_\alpha : \kappa \to V$ can be chosen in V so $[\![[f_\alpha]_N = j(\rho_\alpha)]\!] = \mathtt{1}$. But $[\![[\rho_\alpha]_N = j(\rho_\alpha)(\kappa)]\!] = \mathtt{1}$.

Letting $g : \kappa \to V$ be the function in V given by $g(\beta) = \langle f_\alpha(\beta) : \alpha < \beta \rangle$ for all $\beta < \kappa$ then, in $V[G]$,

$$[g]_N = j(g)(\kappa) = \langle j(f_\alpha)(\kappa) : \alpha < \kappa \rangle = \langle [f_\alpha]_N : \alpha < \kappa \rangle.$$

Hence, ${}^\kappa N \subseteq N$. In particular, $\mathcal{P}^{V[G]}(\kappa) \subseteq N$.

Remark 1.22. Suppose $\mathsf{RVM}(\mathfrak{c})$ and ν is a witness such that $\mathcal{P}(\mathfrak{c})/\mathcal{N}_\nu$ is homogeneous. As mentioned above, it follows that $\mathcal{P}(\mathfrak{c})/\mathcal{N}_\nu \cong \mathtt{Random}_\lambda$ for some $\lambda \geq \omega_1$. It is a result of Gitik and Shelah that in fact $\lambda = 2^{\mathfrak{c}}$, see [22, Theorem 1.1].

Solovay's characterization allows for easy proofs of several results of the classical theory of real-valued measurability. For example:

Corollary 1.23 (Solovay [38]). *If* $\mathsf{RVM}(\kappa)$ *then there is a witnessing probability* ν *such that* $\mathcal{N}_\nu$ (*see Definition* 1.3) *is a normal ideal.*

Proof. Suppose $\mathsf{RVM}(\kappa)$. Let λ be such that in $V^{\mathtt{Random}_\lambda}$ there is $j : V \to N$ with $\mathrm{cp}(j) = \kappa$, and define ν as in the proof of Theorem 1.6. Then $\mathcal{N}_\nu$ is a normal ideal: Suppose $\langle A_\alpha : \alpha < \kappa \rangle$ is a sequence of subsets of κ such that $[\![\kappa \in j(A_\alpha)]\!] = \mathtt{0}$ for all α. Then certainly $[\![\exists \alpha < \kappa\, (\kappa \in j(A_\alpha))]\!] = \mathtt{0}$, i.e., $[\![\kappa \in j\big(\triangledown_{\alpha<\kappa} A_\alpha\big)]\!] = \mathtt{0}$. $\square$

Corollary 1.24 (Solovay [38]**).** *If* $\mathsf{RVM}(\kappa)$ *then* κ *is weakly Mahlo, the weakly Mahlo cardinals are stationary below* κ*, etc.*

Proof. Suppose $\mathsf{RVM}(\kappa)$. Let λ, j, ν be as in the proof of Theorem 1.6. If κ is not weakly Mahlo, then $A = \{\, \alpha < \kappa : \mathrm{cf}(\alpha) < \alpha \,\}$ contains a club in κ and therefore $[\![\kappa \in j(A)]\!] = \mathtt{1}$, i.e., $\mathrm{cf}(\kappa) < \kappa$ in $V^{\mathtt{Random}_\lambda}$. But this is impossible, since $\mathtt{Random}_\lambda$ is ccc. The same argument shows that the non-weakly Mahlo cardinals are in $\mathcal{N}_\nu$, etc. □

Corollary 1.25 (Silver, see [26, Proposition 7.12]**).** *If* $\mathsf{RVM}(\kappa)$ *then the tree property holds for* κ*.*

Proof. Suppose $\mathsf{RVM}(\kappa)$, and let ν be a witnessing probability. Suppose $\mathcal{T}$ is a κ-tree. Without loss, $\mathcal{T} = (\kappa, <_{\mathcal{T}})$. As usual, we will identify $\mathcal{T}$ and its levels $\mathcal{T}_\alpha$, $\alpha < \kappa$, with the underlying subsets of κ. Our convention is that trees grow upward, so if $\mathtt{0}$ is the root of $\mathcal{T}$, $\mathtt{0} <_{\mathcal{T}} a$ for any other $a \in \mathcal{T}$, etc. Let λ be such that in $V^{\mathtt{Random}_\lambda}$ there is $j : V \to N$ with $\mathrm{cp}(j) = \kappa$. Work in $V^{\mathtt{Random}_\lambda}$.

Then $j(\mathcal{T}) \restriction \kappa = \mathcal{T}$. Let $\mu = j(\nu)$, so μ witnesses $\mathsf{RVM}(j(\kappa))$ inside N. For $\alpha < j(\kappa)$, let $A_\alpha = \{\, \beta : \alpha <_{j(\mathcal{T})} \beta \,\}$. Since μ is $j(\kappa)$-complete, $\mu(\mathcal{T}) = 0$ and there is some $\alpha \in j(\mathcal{T})_\kappa$ such that $\mu(A_\alpha) > 0$.

Let $b = \{\, \beta \in \mathcal{T} : \beta <_{j(\mathcal{T})} \alpha \,\}$, and let $\langle\, b_\gamma : \gamma < \kappa \,\rangle$ be its $<_{\mathcal{T}}$-increasing enumeration. Then $\mu(A_{b_\gamma}) \geq \mu(A_{b_\rho})$ whenever $\gamma < \rho$. Since $\kappa > \omega$, for some $\rho < \kappa$ we must have $\mu(A_{b_\rho}) = \mu(A_{b_\tau})$ for all $\tau > \rho$.

For $\beta < \kappa$, $b_\rho <_{\mathcal{T}} \beta$, let $B_\beta = \{\, \gamma < \kappa : \beta <_{\mathcal{T}} \gamma \,\}$. Notice that $\mu(A_\beta) = j(\nu(B_\beta)) = \nu(B_\beta)$ for any such β. Let $\varepsilon = \nu(B_{b_\rho})$. Then $\forall \beta \;{}_{\mathcal{T}}\!> b_\rho$, either $\nu(B_\beta) = \varepsilon$, or $\nu(B_\beta) = 0$ (If $0 < \nu(B_\beta) < \varepsilon$, and $\beta \in \mathcal{T}_\gamma$, then $\beta \neq b_\gamma$ and $B_\beta \cap B_{b_\gamma} = \emptyset$. But then $\nu(B_{b_\gamma}) \leq \nu(B_{b_\rho} \setminus B_\beta) < \varepsilon$, a contradiction.)

Let $\underline{b} = \{\, \beta : \beta \leq_{\mathcal{T}} b_\rho \text{ or } (b_\rho <_{\mathcal{T}} \beta \text{ and } \nu(B_\beta) = \varepsilon) \,\}$. Then $b = \underline{b} \in V$ is a κ-branch through $\mathcal{T}$. □

Stripping away the fat from the above argument allows us to weaken the hypothesis of Corollary 1.25 to the existence of a λ-saturated ideal on κ for some $\lambda < \kappa$ (see [26, Proposition 16.4]).

Corollary 1.26 (Kunen, see [19, Theorem 5N]**).** *If* $\mathsf{RVM}(\mathfrak{c})$ *then* $\diamondsuit_{\mathfrak{c}}$ *holds.* □

This follows from applying to the context of real-valued measurability the standard proof of $\diamondsuit_\kappa$ for κ measurable, we leave the details to the interested reader.

The main result on preservation of real-valued measurability is the following.

Fact 1.27 (Solovay [38, Theorem 7]**).** *Suppose* $\mathsf{RVM}(\kappa)$*. Then* κ *stays real-valued measurable after forcing with any* $\mathtt{Random}_\lambda$ *or more generally* (*by Maharam's theorem*)*, with any measure algebra.* □

Remark 1.28. I do not know if Solovay's characterization allows for an 'elementary embeddings' proof of Fact 1.27: If $\mathsf{RVM}(\kappa)$ and λ, j, N are as in the proof of

Theorem 1.6, so $V^{\mathtt{Random}_\lambda} \models j : V \xrightarrow{\prec} N$, $\mathrm{cp}(j) = \kappa$, then if γ is an ordinal such that $[\![\gamma > j(\kappa)]\!] = \mathtt{1}$, say, it is not clear how to lift j to an embedding

$$\hat{\jmath} : V^{\mathtt{Random}_\gamma} \xrightarrow{\prec} N^{\mathtt{Random}_{j(\gamma)}}$$

in $V^{\mathtt{Random}_\lambda * \dot{\mathtt{Random}}_\mu}$ for some appropriate μ, which seems to be the natural way using elementary embeddings of arguing about Fact 1.27. Even if this is possible, Solovay's original argument from [38] would not be superseded; for example, Solovay's argument indicates natural ways in which new measures can be produced from the ones witnessing $\mathsf{RVM}(\kappa)$. For more on this approach, consider Kunen's proof that $\mathsf{RVM}(\kappa)$ implies the partition relation $\kappa \to (\kappa, \lambda)^2$ for any $\lambda < \omega_1$. See [18] for this argument.

1.1. Absolutely ccc forcing

We now argue that if $\mathbb{P}$ is ccc and $\mathbb{F} = \mathtt{Random}_\lambda$ for some λ, then $\mathbb{P}$ is still ccc in $V^{\mathbb{F}}$.

Definition 1.29. $\mathbb{Q} \in V$ is *absolutely ccc* iff for all outer models $W \supseteq V$, $W \models \mathbb{Q}$ is ccc.[4]

For example $\mathtt{Coll}(\omega, < \omega_1)$, $\mathtt{Add}(\omega, 1)$ (the forcing for adding one Cohen real), and any σ-centered poset are absolutely ccc. The class of absolutely ccc posets is closed under finite support products and finite support iterations[5]. The following example is slightly more interesting, and we will have several occasions to use it.

Claim 1.30. *All measure algebras, in particular all* $\mathtt{Random}_\lambda$*, are absolutely ccc.*

Proof. Let $\mathbb{P} = (\mathbb{B}, \nu) \in V$ be a measure algebra, and let $W \supseteq V$ be an outer model. Let $\omega_1 = \omega_1^W$.

Suppose in W that $\langle b_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-antichain in $\mathbb{B} \setminus \{\mathtt{0}\}$. Then we can assume that for some $n > 0$, $\nu(b_\alpha) > 1/n$ for all α. This is a contradiction: For any $N \in \mathbb{N}$ the sequence $\langle b_m : m < N \rangle$ is in V and since the b_α form an antichain, we have that $\nu\big(\sum^{\mathbb{B}}_{m<N} b_m\big) = \sum_{m<N} \nu(b_m) > \frac{N}{n} > 1$ if N is sufficiently large. ☐

Claim 1.31. *If* $\mathbb{P}$ *is ccc and* $\mathbb{Q}$ *is absolutely ccc, then* $V^{\mathbb{Q}} \models \mathbb{P}$ is ccc.

Proof. Since $\mathbb{P} \times \mathbb{Q} \cong \mathbb{P} * \check{\mathbb{Q}} \cong \mathbb{Q} * \check{\mathbb{P}}$, it suffices to see that $V^{\mathbb{P}} \models \mathbb{Q}$ is ccc, but this holds by hypothesis. ☐

Corollary 1.32. *Let* $\mathbb{F} = \mathtt{Random}_\lambda$ *and let* $\mathbb{P}$ *be ccc. Then* $\mathbb{P}$ *is ccc in* $V^{\mathbb{F}}$*.* ☐

Corollary 1.33. *The existence of atomlessly measurable cardinals is independent of the existence of Suslin trees.*

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[4]A possible metatheory in which this definition takes place is Morse-Kelley. For a ZFC rendering, restrict the outer models to those of the form $V^{\mathbb{F}}$ for $\mathbb{F} \in V$ a poset.

[5]For products, this follows from [31, Theorem II.1.9]. Since the finite support iteration of ccc posets is again ccc, the result for iterations follows easily from the definition of absolutely ccc, because if $\mathbb{P} \in V$ is the finite support iteration of a family $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$, then in any outer model $W \supseteq V$, $\mathbb{P}$ densely embeds into the finite support iteration (in the sense of W) of $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$.

Proof. Let κ be measurable, and suppose S is a Suslin tree. Then

$$1 \Vdash_{\texttt{Random}_\kappa} \mathsf{RVM}(\mathfrak{c}) \text{ and } S \text{ is ccc},$$

by Corollary 1.32. Thus, $V^{\texttt{Random}_\kappa} \models \text{There is a Suslin tree.}$

The other direction is immediate from a result of Laver (see [8, Theorem 3.2.31].) Namely, if $\mathsf{MA}_{\aleph_1}$ holds then for any κ,

$$V^{\texttt{Random}_\kappa} \models \text{Every Aronszajn tree is special.}$$

In particular, if κ is measurable and MA holds, then $V^{\texttt{Random}_\kappa}$ is a model of $\mathsf{RVM}(\mathfrak{c})$ where there are no Suslin trees. □

More interesting consequences of the fact that $\texttt{Random}_\lambda$ is absolutely ccc are explored throughout the paper.

Stronger versions of the following theorem can be obtained, but this suffices for our purposes. Notice the particular case where κ is measurable, so $\mathbb{B}_\nu$ is trivial and $G \in V$.

Theorem 1.34. *Suppose* $\mathsf{RVM}(\kappa)$ *and let* ν *be a witnessing probability such that* $\mathbb{B}_\nu = \mathcal{P}(\kappa)/\mathcal{N}_\nu$ *is homogeneous. Let* G *be* $\mathbb{B}_\nu$*-generic over* V*, and in* $V[G]$ *let* $j : V \to N$ *be the associated generic embedding. Then the forcing* $j(\texttt{Random}_\kappa)/\texttt{Random}_\kappa$ *is isomorphic in* $V[G][H]$ *to* $\texttt{Random}_{j(\kappa)}$*, where* H *is* $\texttt{Random}_\kappa$*-generic over* $V[G]$*.*

Proof. Start by noticing that $(\texttt{Random}_\kappa)^V \in N$. In N,

$$j(\texttt{Random}_\kappa) = \texttt{Random}_{j(\kappa)},$$

so $\texttt{Random}_\kappa \lessdot j(\texttt{Random}_\kappa)$, and the quotient forcing makes sense. Let $\underline{H}$ be the canonical $\texttt{Random}_\kappa$ name for the generic filter and recall that, by definition, $j(\texttt{Random}_\kappa)/\texttt{Random}_\kappa$ is (a $\texttt{Random}_\kappa$ name for) the forcing

$$\mathbb{P} = \{\, q \in j(\texttt{Random}_\kappa) : q \text{ is compatible with every } p \in \underline{H} \,\}.$$

Consequently, fix H a $\texttt{Random}_\kappa$ generic over $V[G]$ and therefore over N, and work in $V[G][H]$.

- *In* $N[H]$, $\mathbb{P} \cong \texttt{Random}_{j(\kappa)}$.
  By Fact 1.18.
- *In* $V[G][H]$, $\mathbb{P}$ *is a* σ*-complete homogeneous boolean algebra.*
  Recall that ${}^\omega N \subset N$, and therefore (by the ccc of $\texttt{Random}_\kappa$) ${}^\omega N[H] \subset N[H]$, from which σ-completeness in $V[G][H]$ follows. Homogeneity is clear, since $\mathbb{P}$ is already homogeneous in $N[H]$.
- *In* $V[G][H]$, $\mathbb{P}$ *is a complete measure algebra.*
  The 'probability measure' witnessing $\mathbb{P}$ is a measure algebra in $N[H]$ is a 'probability measure' in $V[G][H]$, since $N[H]$ is closed under ω-sequences. Hence, $\mathbb{P}$ is a measure algebra. It is ccc, by Claim 1.30. Completeness follows.
- *In* $V[G][H]$, $\mathbb{P}$ *is isomorphic to some* $\texttt{Random}_\rho$ *and, in fact,*

$$\mathbb{P} \cong \texttt{Random}_{|j(\kappa)|}.$$

This follows now from Maharam's theorem. This completes the proof. □

For a generalization, see the first paragraph of the proof of Claim 3.5.

Theorem 1.34 will prove useful in the following sections, where we obtain the consistency of a third-order definable well-ordering of $\mathbb{R}$ together with real-valued measurability of the continuum. That we cannot improve the complexity of this well-ordering in a straightforward fashion is the content of Theorem 2.5 below.

2. Third-order definability

Recall that $\mathsf{HOD}_{\mathbb{R}}$ denotes the class of sets hereditarily ordinal definable using the elements of $\mathbb{R}$ as parameters. $\mathtt{Add}(\kappa, \lambda)$ is the standard forcing for adding λ many Cohen subsets of κ.

Lemma 2.1. *Let G be $\mathbb{F}$-generic over V, where $\mathbb{F} = \mathtt{Add}(\omega, \lambda)$ or $\mathbb{F} = \mathtt{Random}_\lambda$, $\lambda \geq \omega_1$. Let $\mathbb{R} = \mathbb{R}^{V[G]}$. Then, in $V[G]$, there is $\mathbb{R}_0 \subset \mathbb{R}$ and a nontrivial elementary embedding $j : \mathsf{HOD}_{\mathbb{R}_0} \xrightarrow{\prec} \mathsf{HOD}_{\mathbb{R}}$ such that $j \restriction \mathrm{ORD} = \mathrm{id}$.*

Corollary 2.2. *Let $\mathbb{F} = \mathtt{Add}(\omega, \lambda)$ or $\mathbb{F} = \mathtt{Random}_\lambda$ where $\lambda \geq \omega_1$. Then in $V^{\mathbb{F}}$, $\mathsf{HOD}_{\mathbb{R}} \models \neg\mathsf{AC}$ and therefore no relation in $\mathsf{HOD}_{\mathbb{R}}$ defines a well-ordering of $\mathbb{R}$. In particular, $V^{\mathbb{F}} \models L(\mathbb{R}) \models \neg\mathsf{AC}$.*

Proof. In $V^{\mathbb{F}}$ there is a transitive class $N \neq \mathsf{HOD}_{\mathbb{R}}$ and an elementary embedding $j : N \xrightarrow{\prec} \mathsf{HOD}_{\mathbb{R}}$ that does not move the ordinals. It follows from [26, Proposition 5.1] that the Axiom of Choice fails in N and therefore in $\mathsf{HOD}_{\mathbb{R}}$. Since a well-ordering of $\mathbb{R}$ in $\mathsf{HOD}_{\mathbb{R}}$ would induce a well-ordering of $\mathsf{HOD}_{\mathbb{R}}$ in $\mathsf{HOD}_{\mathbb{R}}$, the result follows. □

Remark 2.3. Corollary 2.2 is known, although the proof presented here seems to be new. See for example [31, Exercises VII.E].

Proof of Lemma 2.1. Let G be $\mathbb{F}$-generic over V, where $\mathbb{F}$ is as in the statement of the lemma. By standard arguments (by Maharam's Theorem 1.15 for the case $\mathbb{F} = \mathtt{Random}_\lambda$), $G \cong G_0 \times G_1$, where G_0 is $\mathbb{F}$-generic over V and G_1 is $\mathbb{F}^{V[G_0]}$-generic over $V[G_0]$. Let $\mathbb{R}_0 = \mathbb{R}^{V[G_0]}$ and $\mathbb{R}_1 = \mathbb{R}^{V[G]}$. In $V[G]$ we define a nontrivial $j : \mathsf{HOD}_{\mathbb{R}_0} \xrightarrow{\prec} \mathsf{HOD}_{\mathbb{R}_1}$ such that $j \restriction \mathrm{ORD} = \mathrm{id}$.

For this, notice that any $x \in \mathsf{HOD}_{\mathbb{R}_i}$, $i = 0, 1$, has the form $\tau(\vec{r}, \vec{\alpha})$ where $\vec{r} \in \mathbb{R}_i$, $\vec{\alpha} \in \mathrm{ORD}$, and τ is some term in the language of $\mathsf{HOD}_{\mathbb{R}}$.[6]

Define j by

$$j\big(\tau(\vec{r}, \vec{\alpha})^{\mathsf{HOD}_{\mathbb{R}_0}}\big) = \tau(\vec{r}, \vec{\alpha})^{\mathsf{HOD}_{\mathbb{R}_1}}.$$

We claim j works.

[6]This language expands the language of set theory by closing under *weak Skolem functions*, i.e., those giving definable terms, so for $\varphi(x, y)$ a formula and z a set, $\tau_\varphi(z)$ is defined iff $\exists! x\, \varphi(x, z)$, and $\tau_\varphi(z) = u$ iff $\varphi(u, z)$. We cannot simply use (definable) Skolem functions, since AC fails in $\mathsf{HOD}_{\mathbb{R}}$. If the reader does not want to bother formalizing this language, it suffices that for every $x \in \mathsf{HOD}_{\mathbb{R}}$ there is a formula $\phi(v_1, \vec{v}_2, \vec{v}_3)$ in the language of set theory, and there are reals $\vec{r}$ and ordinals $\vec{\alpha}$ such that

$$\mathsf{HOD}_{\mathbb{R}} \models x = \{\, y : \phi(y, \vec{r}, \vec{\alpha}) \,\}. \qquad (\dagger)$$

The reader should have no problem using (†) to translate our use of terms into standard notation.

Let $\varphi(v_0, \ldots, v_n)$ be any formula, let $\tau_0(\vec{v}_0, \vec{v}_1), \ldots, \tau_n(\vec{v}_0, \vec{v}_1)$ be terms, and let $x_0, \ldots, x_n \in \mathsf{HOD}_{\mathbb{R}_0}$ be given by $x_i = \tau_i(\vec{r}_i, \vec{\alpha}_i)^{\mathsf{HOD}_{\mathbb{R}_0}}$. By composing each τ_i with some projections and some recursive surjections $\pi_0 : \mathbb{R} \to \mathbb{R}^{<\omega}$ and $\pi_1 : \mathrm{ORD} \to \mathrm{ORD}^{<\omega}$, we may assume $\vec{r}_i = r$, $\vec{\alpha}_i = \alpha$ for all i. Let $\psi(v_0, v_1) \equiv \varphi\big(\tau_0(v_0, v_1), \ldots, \tau_n(v_0, v_1)\big)$ and $\mu(v_0, v_1) \equiv \mathsf{HOD}_{\mathbb{R}} \models \psi(v_0, v_1)$.

The whole point of the argument is that there is a set $X \in \mathcal{P}_{\omega_1}(\lambda)$ such that $r \in V[G_0 \restriction X]$, and there are $\mathbb{F}^{V[G_0 \restriction X]}$-generics over $V[G_0 \restriction X]$, $\mathcal{G}_0$ and $\mathcal{G}_1$, such that $V[G_0 \restriction X][\mathcal{G}_0] = V[G_0]$ and $V[G_0 \restriction X][\mathcal{G}_1] = V[G]$.

Then

$$\begin{aligned} \mathsf{HOD}_{\mathbb{R}_0} \models \psi(r, \alpha) &\iff V[G_0] \models \mu(r, \alpha) \\ &\iff \exists p \in \mathcal{G}_0 \big(V[G_0 \restriction X] \models p \Vdash_{\mathbb{F}} \mu(\check{r}, \check{\alpha})\big) \\ &\overset{(*)}{\iff} V[G_0 \restriction X] \models \mathbb{1}_{\mathbb{F}} \Vdash_{\mathbb{F}} \mu(\check{r}, \check{\alpha}) \\ &\overset{(*)}{\iff} \exists q \in \mathcal{G}_1 \big(V[G_0 \restriction X] \models q \Vdash_{\mathbb{F}} \mu(\check{r}, \check{\alpha})\big) \\ &\iff V[G] \models \mu(r, \alpha) \\ &\iff \mathsf{HOD}_{\mathbb{R}_1} \models \psi(r, \alpha), \end{aligned}$$

where $(*)$ holds by the weak homogeneity of $\mathbb{F}$. Recall that a forcing $\mathbb{P}$ is *weakly homogeneous* (see [26, before Proposition 10.19]) iff for all $p, q \in \mathbb{P}$ there is an automorphism π of $\mathbb{P}$ such that $\pi(p)$ is compatible with q. Clearly, $\mathbb{F}$ is weakly homogeneous. It is a basic result in the theory of forcing ([26, Proposition 10.19]) that if $\mathbb{P}$ is weakly homogeneous, $\phi(v_1, \ldots, v_n)$ is a formula in the forcing language, all of its free variables displayed, and $x_1, \ldots, x_n \in V$, then either $\mathbb{1} \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \ldots, \check{x}_n)$ or else $\mathbb{1} \Vdash_{\mathbb{P}} \neg\varphi(\check{x}_1, \ldots, \check{x}_n)$.

The chain of equivalences shown above implies immediately that j is well defined and elementary. By definition, $j \restriction \mathrm{ORD} = \mathrm{id}$, and we are done. □

Remark 2.4. Notice that with the same notation as above,

$$j \restriction L(\mathbb{R}_0) : L(\mathbb{R}_0) \xrightarrow{\prec} L(\mathbb{R}_1).$$

Essentially the same argument shows that if $\omega_1 \le \lambda_1 \le \lambda_2$, H_1 is $\mathtt{Random}_{\lambda_1}$- (respectively, $\mathtt{Add}(\omega, \lambda_1)$-) generic over V, and H_2 is $\mathtt{Random}_{\lambda_2}$- (respectively, $\mathtt{Add}(\omega, \lambda_2)$-) generic over $V[H_1]$, then in $V[H_1][H_2]$ there is a nontrivial embedding

$$j : L(\mathbb{R}^{V[H_1]}) \xrightarrow{\prec} L(\mathbb{R}^{V[H_1][H_2]})$$

such that $j \restriction \mathrm{ORD} = \mathrm{id}$. To see this, it suffices to argue that if $\varphi(x, y)$ is a formula, r is a real, α is an ordinal, and $\dot{\mathbb{R}}$ is a term (for the appropriate forcing) for the reals of the generic extension, then $\mathbb{1} \Vdash_{\mathtt{Random}_{\lambda_1}} L(\dot{\mathbb{R}}) \models \varphi(r, \alpha)$ iff $\mathbb{1} \Vdash_{\mathtt{Random}_{\lambda_2}} L(\dot{\mathbb{R}}) \models \varphi(r, \alpha)$.

Suppose $|\lambda_1| < |\lambda_2|$, and let $\mathbb{P}$ be the forcing for collapsing λ_2 to λ_1 with countable conditions, so $\mathbb{P}$ does not add any reals. By Fact 1.19, if $\mathcal{G}$ is $\mathtt{Random}_{\lambda_1}$-generic over $V^{\mathbb{P}}$ then $\mathcal{G}$ is $\mathtt{Random}_{\lambda_1}$-generic over V. The same holds for $\mathtt{Random}_{\lambda_2}$-generic filters, and we are done by weak homogeneity: In $V^{\mathbb{P}}$, $\mathtt{Random}_{\lambda_1}$ and

$\texttt{Random}_{\lambda_2}$ are equivalent. Let H be $\mathbb{P}$-generic over V and let G be $\texttt{Random}_{\lambda_1}$-generic over $V[H]$. Then the reals of $V[H][G]$ and the reals of $V[G]$ coincide. But G is also $\texttt{Random}_{\lambda_2}$-generic over $V[H]$.

Let r be a real in $V[G]$. Recall that $\texttt{Random}_\lambda$ and $\texttt{Random}_\omega * \texttt{Random}_\lambda$ are isomorphic for any cardinal λ, by Maharam's Theorem 1.15, so we may write $G \cong G_0 \times G_1$ where $r \in V[G_0]$, G_0 is generic over V for a forcing isomorphic to $\texttt{Random}_\omega$, and G_1 is generic over $V[G_0]$ for a forcing isomorphic to $\texttt{Random}_{\lambda_1}$. Let α be an ordinal and let φ be a formula. It follows that

$$\begin{aligned} V[H][G] \models L(\mathbb{R}) \models \varphi(r,\alpha) &\iff V[G] \models L(\mathbb{R}) \models \varphi(r,\alpha) \\ &\iff \exists p \in G_1 \left(V[G_0] \models p \Vdash_{\texttt{Random}_{\lambda_1}} \varphi(\check{r},\check{\alpha})\right) \\ &\iff V[G_0] \models \mathbb{1} \Vdash_{\texttt{Random}_{\lambda_1}} \varphi(\check{r},\check{\alpha}), \end{aligned}$$

and exactly the same argument with $\texttt{Random}_{\lambda_2}$ instead of $\texttt{Random}_{\lambda_1}$ shows that

$$V[H][G] \models L(\mathbb{R}) \models \varphi(r,\alpha) \iff V[G_0] \models \mathbb{1} \Vdash_{\texttt{Random}_{\lambda_2}} \varphi(\check{r},\check{\alpha})$$

and, therefore,

$$V[G_0] \models \mathbb{1} \Vdash_{\texttt{Random}_{\lambda_1}} \varphi(\check{r},\check{\alpha}) \iff V[G_0] \models \mathbb{1} \Vdash_{\texttt{Random}_{\lambda_2}} \varphi(\check{r},\check{\alpha}),$$

as we needed to show (notice we can ignore G_0 if $r \in V$).

The argument for Cohen forcing is identical.

Theorem 2.5. *If $\kappa \leq \mathfrak{c}$ and $\mathsf{RVM}(\kappa)$ then no well-ordering of $\mathbb{R}$ belongs to $L(\mathbb{R})$.*

Proof. The argument is standard. Assume by contradiction that $\mathsf{RVM}(\kappa)$ and there is $\varphi(x,y,z,w)$ a formula in the language of $L(\mathbb{R})$ such that for some real t and ordinal α, the relation between reals

$$r < s \iff L(\mathbb{R}) \models \varphi(r,s,t,\alpha)$$

is a well-ordering of $\mathbb{R}$. The least such α is definable in $L(\mathbb{R})$, so there is such a formula φ' all of whose parameters are reals. Let λ be as above, so in $V^{\texttt{Random}_\lambda}$ there is an embedding $j : V \xrightarrow{\prec} N$ such that $\mathrm{cp}(j) = \kappa$ and ${}^\omega N \subseteq N$. Then

$$j \restriction L(\mathbb{R})^V : L(\mathbb{R})^V \xrightarrow{\prec} L(\mathbb{R})^{V^{\texttt{Random}_\lambda}}.$$

In particular, there is $t \in \mathbb{R}^V$ such that $\varphi'(x,y,t)$ still defines a well-ordering of $\mathbb{R}^{V^{\texttt{Random}_\lambda}}$. This is impossible by Lemma 2.1 because $\lambda \geq \omega_1$. □

It follows in particular that no projective (i.e., second-order in the language of arithmetic) formula defines a well-ordering of the reals if there are atomless real-valued measurable cardinals.

Definition 2.6. A Σ^2_n formula is a formula ψ over a three-sorted structure of the form

$$(\mathcal{P}(\mathcal{P}(\mathbb{N})), \mathcal{P}(\mathbb{N}), \mathbb{N}, \in, \dots)$$

such that

$$\psi \equiv \exists X_1 \subseteq \mathcal{P}(\mathbb{N})\, \forall X_2 \subseteq \mathcal{P}(\mathbb{N}) \dots \varphi(X_1, X_2, \dots),$$

where there are n alternations of quantifiers over subsets of $\mathcal{P}(\mathbb{N})$, and φ is a projective statement, i.e., it only involves quantification over $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$.

It is standard to refer to a Σ^2_n statement as being *third-order* (in the language of arithmetic); similarly, a projective statement is usually called second-order (in the language of arithmetic). An equivalent formulation is mentioned below, in Remark 2.8.

We define Π^2_n, Δ^2_n as usual: A statement is Π^2_n iff its negation is Σ^2_n and it is Δ^2_n iff it is simultaneously equivalent to Σ^2_n and Π^2_n statements. Notice that if a linear ordering of $\mathbb{R}$ is Σ^2_n or Π^2_n, then it is automatically Δ^2_n: Suppose $\phi(x, y)$ is a Σ^2_n formula defining a linear ordering. Then $\neg\phi(r, s)$ iff $s = r$ or $\phi(s, r)$.

We close this section with a fact that (we hope) helps to understand the form taken by the well-orderings obtained in the following sections. The point is that we want to codify definability computations in the language of set theory within the language of third-order arithmetic.

We state the fact in a somewhat informal manner, to emphasize its flexibility. Here, $\mathsf{ZFC}^{-\varepsilon}$ is a sufficiently strong fragment of ZFC. For a specific version, we can take $\mathsf{ZFC}^{-\varepsilon}$ to mean $\mathsf{ZFC}^{-} + \mathcal{P}(\mathbb{R})$ exists (considering a large H_η instead of V_η in the proof below), or $\mathsf{ZFC}{\restriction}\Sigma_{200}$, i.e., ZFC with replacement restricted to Σ_{200} statements.

Fact 2.7. *Let $\varphi(\vec{x})$ be a Σ^2_1-formula. Then there is ψ, and a transitive structure $M \models \mathsf{ZFC}^{-\varepsilon}$ such that $\mathbb{R} \subseteq M$, $|M| = \mathfrak{c}$, or even ${}^{\omega}M \subseteq M$, such that for all reals $\vec{r}$,*

$$\varphi(\vec{r}) \iff M \models \psi(\vec{r}).$$

Proof. The existence of such an M is easily seen to be equivalent to a Σ^2_1-formula. Conversely, given φ, let η be large enough, so for any $\vec{r}$,

$$\varphi(\vec{r}) \iff V_\eta \models (\mathcal{P}(\mathbb{R}), \mathbb{R}, \omega, \dots) \models \varphi(\vec{r}),$$

and we can take as M a suitable substructure of V_η. $\square$

Remark 2.8. In fact, the pointclass Σ^2_n can be identified by this method with the class $\Sigma_n(H_{\mathfrak{c}^+}, \in, H_{\omega_1}, H_\omega)$, where H_{ω_1} and H_ω are seen as parameters and therefore quantification over them is considered bounded.

Fact 2.7 and Remark 2.8 motivate the general structure of the constructions that produce Σ^2_n-well-orderings: A model needs to be produced satisfying certain first-order property ψ (somehow related to properties of the surrounding universe). Since the model can resemble the first-order theory of the surrounding universe as much as necessary, the need to satisfy ψ is in general not the main problem and is expected in practice to be achieved by forcing. The difficulty arises in trying to isolate the model or models that we have in mind from possibly fake ones, which can be thought of as proving a "correctness" theorem. This suggests the need to establish some kind of "thinness" condition, usually in tension with the width the forcing extension provides, this being in practice the main source of complications when implementing this strategy. This general framework will be illustrated with the results of this paper.

3. Σ^2_2-well-orderings

We begin with a construction based on a technique which goes back at least to Woodin's work on generalized Prikry forcing. Starting with a measurable cardinal, this technique produces a model where the cardinal is real-valued measurable, and the generic codes a subset of the reals. This construction is a prototype of several arguments showing the consistency of $\mathsf{RVM}(\mathfrak{c})$ with different kinds of definable well-orderings, and we illustrate some of them. In section 6 we show how, working over $L[\mu]$, a variation due to Woodin of the construction given in this section establishes the consistency of real-valued measurability of the continuum together with a Δ^2_1-well-ordering of $\mathbb{R}$. Here we obtain the consistency of $\mathsf{RVM}(\mathfrak{c})$ together with a Δ^2_2-well-ordering of $\mathbb{R}$ without any restrictions in the large cardinal structure of the universe. The combinatorial tool we use to carry out our coding was first considered in [3], in the presence of MA.

Theorem 3.1. *If κ is measurable in V and $2^\kappa = \kappa^+$, then there is a forcing $\mathbb{F}$ of size κ such that*

$$1 \Vdash_{\mathbb{F}} \mathfrak{c} = \kappa,\ \mathsf{RVM}(\mathfrak{c}),\ \text{and there is a } \Delta^2_2 \text{ well-ordering of } \mathbb{R}.$$

Proof. By a preliminary forcing, if necessary, we may assume GCH holds below κ (see for example [25]).

Let $\mathbb{Q} = \mathtt{Random}_\kappa$. Let $\mathbb{P}$ be the Easton product over the inaccessible cardinals $\lambda < \kappa$ of $\prod_{n\in\omega} \mathtt{Add}(\lambda^{+1+3n}, \lambda^{+3+3n})$, where the product is inverse (i.e., fully supported). Let $\mathbb{S} = \mathbb{P} \times \mathbb{Q}$, and let $G_{\mathbb{P}} \times G_{\mathbb{Q}}$ be $\mathbb{S}$-generic over V.

The proof rests on a "lifting" argument, which we isolate as follows:

Claim 3.2. *Let $j : V \to N$ be an ultrapower embedding by a normal measure on κ. Then $j(\mathbb{Q})/\mathbb{Q}$ is isomorphic to an appropriate random forcing in any intermediate model between $V[G_{\mathbb{Q}}]$ and $V_1 := V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$, inclusive, i.e., for any such model M there is a λ such that*

$$M \models j(\mathbb{Q})/\mathbb{Q} \cong \mathtt{Random}_\lambda.$$

There is $G^* \in V$ such that:

- *If H is $j(\mathbb{Q})/\mathbb{Q}$-generic over V_1 then, in $V_1[H]$, j lifts to*

$$j_2 : V_1 \to N[G_{\mathbb{P}}][G^*][G_{\mathbb{Q}}][H]$$

  and therefore (by Solovay's Theorem 1.6) $\mathsf{RVM}(\mathfrak{c})$ holds in V_1.
- *The restriction of j_2 to $V[G_{\mathbb{P}}]$ is $j_1 : V[G_{\mathbb{P}}] \to N[G_{\mathbb{P}}][G^*]$ (so κ remains measurable in $V[G_{\mathbb{P}}]$). There is a forcing $\mathbb{P}_{tail} \in N$ such that $j(\mathbb{P}) = \mathbb{P} \times \mathbb{P}_{tail}$, and G^* is $\mathbb{P}_{tail}$-generic over N.*

Similarly, the restriction of j_2 to $V[G_{\mathbb{Q}}][H]$ is $j_3 : V[G_{\mathbb{Q}}] \to N[G_{\mathbb{Q}}][H]$ and witnesses $\mathsf{RVM}(\mathfrak{c})$ in $V[G_{\mathbb{Q}}]$. Finally, $\mathbb{R}^{V[G_{\mathbb{Q}}]} = \mathbb{R}^{V[G_{\mathbb{Q}}][G_{\mathbb{P}}]}$.

Proof. We begin by showing:

Subclaim 3.3. *$\mathbb{P}$ preserves the measurability of κ. In fact, there is $G^* \in V$ such that whenever G is $\mathbb{P}$-generic over V, $G \times G^*$ is $j(\mathbb{P})$-generic over N, and we can lift j to an embedding $j_1 : V[G] \to N[G \times G^*]$.*

(*Cf.* [23, Lemma 2.2.4] or [13, Fact 3.1].)

Proof. By elementarity, in N, $j(\mathbb{P}) = \mathbb{P} \times \mathbb{P}_{\text{tail}}$ where $\mathbb{P}_{\text{tail}}$ is the Easton product of $\prod_{n\in\omega} \mathtt{Add}(\lambda^{+1+3n}, \lambda^{+3+3n})$, the product being inverse (i.e., fully supported) and λ ranging over the inaccessible cardinals $\lambda \in [\kappa, j(\kappa))$. In N, this set is κ^+-closed. But ${}^{\kappa}N \subset N$, so in fact it is κ^+-closed in V. Now notice that $|\mathcal{P}^N(\mathbb{P}_{\text{tail}})| = |(2^{j(\kappa)})^N| = |j(2^\kappa)| \leq (2^\kappa)^\kappa = 2^\kappa = \kappa^+$, where the last equality holds by hypothesis. Thus, the number of dense subsets of $\mathbb{P}_{\text{tail}}$ which belong to N is at most κ^+, and a straightforward induction lets us build (in V) a decreasing sequence of conditions which meet all of them. The filter G^* they generate is therefore $\mathbb{P}_{\text{tail}}$-generic over N.

It remains to argue that if G is $\mathbb{P}$-generic over V, then $G \times G^*$ is $j(\mathbb{P})$-generic over N, which amounts to showing that G and G^* are mutually generic. If so, j lifts to j_1 in the usual way[7]: For σ a $\mathbb{P}$-name, $j_1(\sigma_G) := j(\sigma)_{G\times G^*}$. The standard argument (see [13, Fact 2.1]) proves that j_1 is well defined and elementary.

But mutual genericity is clear: Since $N[G^*] \subseteq V$, if G is $\mathbb{P}$-generic over V, it is also $\mathbb{P}$-generic over $N[G^*]$.

This completes the proof of Subclaim 3.3. □

Notice that $\mathbb{P}$ is ω_1-closed, so $\mathbb{R}^{V[G_{\mathbb{P}}\times G_{\mathbb{Q}}]} = \mathbb{R}^{V[G_{\mathbb{Q}}]}$.

Subclaim 3.4. *In $V[G_{\mathbb{P}}][G_{\mathbb{Q}}][H]$, j_1 lifts to*

$$j_2 : V[G_{\mathbb{P}}][G_{\mathbb{Q}}] \to N[G_{\mathbb{P}}][G^*][G_{\mathbb{Q}}][H].$$

The restriction of j_2 to $V[G_{\mathbb{Q}}]$ is an embedding

$$j_3 : V[G_{\mathbb{Q}}] \to N[G_{\mathbb{Q}}][H]$$

definable in $V[G_{\mathbb{Q}}][H]$.

Proof. As expected, simply set

$$j_2(\tau_{G_{\mathbb{Q}}}) = j_1(\tau)_{G_{\mathbb{Q}}\frown H},$$

for τ a $\mathbb{Q}$-name in $V[G_{\mathbb{P}}]$. As before, j_2 is well defined and elementary. Since j_1 extends j, $j_3 = j_2 \restriction V[G_{\mathbb{Q}}] : V[G_{\mathbb{Q}}] \to N[G_{\mathbb{Q}}][H]$ is given by $j_3(\tau_{G_{\mathbb{Q}}}) = j(\tau)_{G_{\mathbb{Q}}\frown H}$ for τ a $\mathbb{Q}$-name in V, and is definable in $V[G_{\mathbb{Q}}][H]$ as claimed.

This completes the proof of Subclaim 3.4. □

The proof of Claim 3.2 is complete. □

[7] This is the standard way of showing that if ρ is measurable, then it is still real-valued measurable in $V^{\mathtt{Random}_\rho}$.

In $V[G_{\mathbb{Q}}]$, let $A = \langle r_\alpha : \alpha < \kappa \rangle$ be a well-ordering of $\mathbb{R}$. In $V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$, define g as follows:

Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ enumerate in V the inaccessible cardinals below κ. Let G_α be the part of $G_{\mathbb{P}}$ which is generic for $\prod_{n\in\omega} \mathtt{Add}(\delta_\alpha^{+1+3n}, \delta_\alpha^{+3+3n})$. Write $G_\alpha \cong \prod_{n\in\omega} G_\alpha(n)$, where $G_\alpha(n)$ is the part of G_α generic for $\mathtt{Add}(\delta_\alpha^{+1+3n}, \delta_\alpha^{+3+3n})$. Then

$$g = \prod_{\alpha<\kappa} G^*_\alpha,$$

where $G^*_\alpha = \prod_{n\in\omega} G^*_\alpha(n)$ and

$$G^*_\alpha(n) = \begin{cases} G_\alpha(n) & \text{if } n \in r_\alpha, \\ 1_{\mathtt{Add}(\delta_\alpha^{+1+3n},\delta_\alpha^{+3+3n})} & \text{if } n \notin r_\alpha. \end{cases}$$

Claim 3.5. *$\kappa = \mathfrak{c}$ stays real-valued measurable in $V[G_{\mathbb{Q}}][g]$.*

Proof. By Theorem 1.34, $j(\mathbb{Q})/\mathbb{Q}$ is isomorphic to $\mathtt{Random}_{j(\kappa)}$ in $V[G_{\mathbb{Q}}]$. It follows that in $V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$ as well as in $V[G_{\mathbb{Q}}][g]$, $j(\mathbb{Q})/\mathbb{Q}$ is still a complete measure algebra, since the forcing for which g is generic is a factor of $\mathbb{P}$, which is ω_2-closed in V and therefore ω_2-dense in $V[G_{\mathbb{Q}}]$ by Easton's lemma, see [13, Fact 4.1]. Since $j(\mathbb{Q})/\mathbb{Q}$ was homogeneous in $V[G_{\mathbb{Q}}]$, it is still homogeneous in $V[G_{\mathbb{Q}}][g]$ and in $V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$. We conclude that $j(\mathbb{Q})/\mathbb{Q}$ is still isomorphic to $\mathtt{Random}_{j(\kappa)}$, by Maharam's Theorem 1.15.

Let H be $j(\mathbb{Q})/\mathbb{Q}$-generic over $V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$. We will show that in $V[G_{\mathbb{Q}}][g][H]$, j lifts to

$$j^* : V[G_{\mathbb{Q}}][g] \to N[j^*(G_{\mathbb{Q}})][j^*(g)].$$

This amounts to defining $j^*(G_{\mathbb{Q}})$ and $j^*(g)$, and checking that the induced map j^* is well defined and elementary. Once this is done, Solovay's Theorem 1.6 implies the claim.

Set $j^*(G_{\mathbb{Q}}) = G_{\mathbb{Q}}{}^\frown H$. To define $j^*(g)$, it suffices to define $j^*(g)_{[\kappa,j(\kappa))}$ (so $j^*(g) = g^\frown j^*(g)_{[\kappa,j(\kappa))}$). The intention is that the definition of $j^*(g)$ copies that of g, so we must start by defining $j^*(A)$.

Since $A \in V[G_{\mathbb{Q}}]$, $j_3(A) \in V[G_{\mathbb{Q}}][H]$. We set $j^*(A) = j_3(A)$ (with j_3, etc, as in Claim 3.2). The key observation is that we do not really need a whole $j(\mathbb{P})$-generic to define $j^*(g)_{[\kappa,j(\kappa))}$, but a $\mathbb{P}_{\text{tail}}$-generic suffices: Remember that G^*, as built in Subclaim 3.3, is in V. We can now set

$$j^*(g)_{[\kappa,j(\kappa))} := \prod_{\alpha\in[\kappa,j(\kappa))} \prod_{n\in\omega} G^{**}_{\alpha,n},$$

where $G^{**}_{\alpha,n}$ is the $\mathtt{Add}(\delta_\alpha^{+1+3n}, \delta_\alpha^{+3+3n})$-generic added by G^* to N, if $n \in j_3(r)_\alpha$, and the trivial condition otherwise.

Here, $\langle \delta_\alpha : \alpha < j(\kappa) \rangle = j(\langle \delta_\alpha : \alpha < \kappa \rangle)$ is the increasing enumeration of the inaccessible cardinals in N below $j(\kappa)$ and $j_3(A) = \langle j_3(r)_\alpha : \alpha < j(\kappa) \rangle$ is the well-ordering of the reals of N .

Extend j^* to a map

$$j^* : V[G_{\mathbb{Q}}][g] \to N[j^*(G_{\mathbb{Q}})][j^*(g)]$$

in the usual way. Notice that j^* is simply the restriction of j_2, as defined in Subclaim 3.4, to $V[G_{\mathbb{Q}}][g]$. This proves j^* is well defined and elementary. Finally, notice that j^* is definable in $V[G_{\mathbb{Q}}][g][H]$. This concludes the proof of Claim 3.5. □

Remark 3.6. The argument just given is quite general. It works as long as $\mathbb{P}$ is a reasonably definable product of sufficiently closed small forcings. The set we called A can code any subset of the reals in $V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$. By coding A inside a "subproduct" g of $G_{\mathbb{P}}$, we avoid having to set up any sort of book-keeping devices in the ground model in order to define the well-ordering alongside the iteration. As a matter of fact, we do not need to worry about defining in the ground model (as an iteration or otherwise) the forcing whose generic is g.

Notice also that, in spite of this generality, some argument was required, since it is not necessarily true that if W is a forcing extension of V preserving $\mathsf{RVM}(\kappa)$, then any intermediate extension $V \subseteq M \subseteq W$ satisfies $\mathsf{RVM}(\kappa)$ as well. This observation (with κ measurable in V and W) is due to Kunen, see [30]; we present in section 4 a proof of this result, different from the argument in [30]. The proof in section 4 uses the technique illustrated in this section: Starting with an embedding $j : V \to N$, we find in V an N-generic filter for a sufficiently closed forcing notion living in N. A proof dealing specifically with atomless measurability has been produced by Gitik, see [22, Theorem 2]; Gitik's proof can be seen as an elaboration of the argument in section 4, and the reader may find it profitable to read section 4 before consulting [22].

Now we continue with the proof of Theorem 3.1. All what remains is to see that we can "decode" the well-ordering A from g in a Σ^2_2-way in $V[G_{\mathbb{Q}}][g]$. The forcing $\mathbb{F}$ is then the factor of $\mathbb{S}$ for which $G_{\mathbb{Q}} \times g$ is a generic.

The key to our coding is the following notion (see [4]):

Definition 3.7. Let λ be regular. The *club base number* for λ is

$$\min\{\, |X| : X \subseteq \mathcal{P}(\lambda) \text{ and } \forall \text{ club } C \subseteq \lambda\, \exists \text{ club } D \in X\, (D \subseteq C)\,\}.$$

So the club base number for λ is the coinitiality of the club filter at λ, ordered under inclusion. Any collection X of club subsets of λ realizing the minimum above generates the club filter at λ by closing under supersets.

If λ is regular and $2^\lambda = \lambda^+$, then the club base number for λ is λ^+, while if λ^{++} Cohen subsets of λ are added, their closures are club sets containing no club from the ground model, and mutual genericity guarantees that the club base number at λ is λ^{++}.

It follows that in $V[g]$ the inaccessible cardinals below κ are just the δ_α, $\alpha < \kappa$, and the club base number for δ_α^{+1+3n} is either δ_α^{+2+3n} or $\delta_\alpha^{+3(n+1)}$ depending on whether $G^*_\alpha(n)$ is trivial or not, since the base number for $\delta_{\alpha_1}^{+1+3n}$ is not affected by forcing with (a subproduct of) $\prod_{m\in\omega} \mathtt{Add}(\delta_{\alpha_2}^{+1+3m}, \delta_{\alpha_2}^{+3+3m})$ for $\alpha_2 \neq \alpha_1$.

Maybe a more detailed argument is in order: Let $\lambda < \kappa$ be inaccessible, let $n < \omega$, and write $\mathbb{P} \cong \mathbb{P}_{\lambda,n} \times \mathtt{Add}(\lambda^{+1+3n}, \lambda^{+3+3n}) \times \mathbb{P}^{\lambda,n}$, where $\mathbb{P}_{\lambda,n}$ corresponds to the factors of $\mathbb{P}$ that add Cohen subsets to cardinals strictly smaller than λ^{+1+3n}, and $\mathbb{P}^{\lambda,n}$ corresponds to those factors that add Cohen subsets to strictly bigger cardinals. Then $\mathbb{P}^{\lambda,n}$ is sufficiently closed that it cannot ("by accident") add a subset of λ^{+1+3n}, while $\mathbb{P}_{\lambda,n}$ satisfies a sufficiently small chain condition that any club subset of λ^{+1+3n} that it adds contains a club in the ground model. Finally, $G_{\mathbb{Q}}$ is added by ccc forcing, so it does not affect any of the club base numbers that concern us. It follows that in $V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$ the only club base numbers that are affected are those that we have explicitly changed by means of $G_{\mathbb{P}}$, and therefore in $V[G_{\mathbb{Q}}][g]$ we have coded A by means of the club base numbers which have been altered.

Now observe that in $V[G_{\mathbb{Q}}][g]$ we can define A, or rather the corresponding order relation $<_A$ on $\mathbb{R}$ as follows:

Let $\Psi(M)$ denote the conjunction of the following requirements:

$M \models \mathsf{ZFC}^-$, M is transitive, $|M| = \mathfrak{c}$, and $\mathbb{R} \subseteq M$. (Notice that this implies $\mathrm{ORD}^M \geq \mathfrak{c}$.) Moreover,

1. M computes cofinalities correctly, that is, if $\lambda, \mu \in M$, and there is $f : \lambda \to \mu$ cofinal, then there is such an $f \in M$.
2. For all $C \subseteq \lambda < \mathfrak{c}$ club, there is $D \in M$, $D \subseteq \lambda$ club, such that $D \subseteq C$.
3. M computes club base numbers correctly, that is, for all $\lambda < \mathfrak{c}$, $\mathcal{F} \subseteq \mathcal{P}(\lambda)^M$ collection of club sets, $|\mathcal{F}| < \mathfrak{c}$, there is $\mathcal{G} \in M$ collection of club sets such that $\mathcal{G}$ is coinitial in $\mathcal{F}$.[8]

Finally, for all $r \in \mathbb{R}$ there is in M a unique sequence of club base numbers starting at a weakly inaccessible[9] which (in the obvious way) code r, and any weakly inaccessible codes some r.

Notice that $\Psi(M)$ is the conjunction of the statement that $|M| = \mathfrak{c}$ and a $\Pi_1(H_{\mathfrak{c}^+}, \in, H_{\omega_1})$ statement about M. Notice as well that M does not have any cardinals above $\mathfrak{c}$. For example, fixing a surjection $\pi : \mathbb{R} \to M$, condition 2 is seen to be Π_1 since the existential quantifier is actually a quantifier over reals, i.e., there is a real r such that, in M, $\pi(r) = D$ is a club set.

For x, y reals, let $\psi(x, y)$ hold iff

There is M such that $\Psi(M)$ holds and in M the sequence coding x appears before the sequence coding y.

Since M is only a model of ZFC^-, $\mathcal{P}(\lambda)^M$ as in 3 above, is to be interpreted as a definable class. This does not affect the desired complexity of ψ.

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[8] The requirement on the size of $\mathcal{F}$ is not essential. We just include it to ensure the universal quantifier in the definition of the well-ordering we obtain actually ranges over bounded subsets of $\mathfrak{c}$.

[9] Since the ground model satisfies GCH, the weakly inaccessible and the strongly inaccessible cardinals coincide here, and we took care of coding reals at each inaccessible cardinal. It is by no means essential that we decide to code using the inaccessible cardinals, and the coding could have occurred at many other places (say, starting at limit cardinals), with only a straightforward variation in the construction above being required.

The relation ψ just defined can be rendered Σ^2_2 in a straightforward fashion. We are done once we verify that $x <_A y$ holds for reals x, y if and only if $\psi(x, y)$ does. That $x <_A y$ implies $\psi(x, y)$ is easy, $M = V[G_{\mathbb{Q}}][g]_\kappa$ is a witness. To see the converse just observe that any M witnessing $\psi(x, y)$ is correct about cofinalities below κ, and computes correctly club base numbers of cardinals below κ. The uniqueness of the coding of reals by club base numbers ensures that no fake codings (witnessing false relations $x <_A y$) may arise, since every inaccessible below κ is assigned some real this way, and this assignment is a bijection; in particular, $\mathrm{ORD}^M = \mathfrak{c}$ must hold (since no real is coded by $\mathfrak{c}$). Since g was defined precisely to code A using the club base numbers, $\psi(x, y)$ implies $x <_A y$. This completes the proof of Theorem 3.1. □

Remark 3.8. Notice that M as above is not a model of "$\mathbb{R}$ exists". $\mathbb{R}$ and the relation ψ are only *definable* over M. Obviously, any transitive model N of enough set theory that contains all the reals and where there is a well-ordering of $\mathbb{R}$ must satisfy $\mathrm{ORD}^N > \mathfrak{c}$.

Let $(\Sigma^2_2)^+$ denote the class of statements about the reals expressible as a Boolean combination of Σ^2_2 statements. As a consequence of the argument above and Solovay's theorem on preservation of real-valued measurability (Fact 1.27) we obtain that generic invariance of $(\Sigma^2_2)^+$ with respect to real-valued measurability of the continuum[10] is not a theorem of ZFC, even in the presence of projective absoluteness. In effect, the fact that ψ defines a well-ordering of $\mathbb{R}$, for ψ as above, can be expressed as a $(\Sigma^2_2)^+$ statement[11], it can be made true over V as long as there are measurable cardinals in V, and can be made false afterwards simply by adding ω_1 many random reals, see Lemma 2.1.

[10]Generic invariance of a class Γ of sentences with respect to a statement ϕ means that whenever $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration of set forcings such that $V^{\mathbb{P}} \models \phi + 1_{\dot{\mathbb{Q}}} \Vdash \phi$, then for all $\psi \in \Gamma$, $V^{\mathbb{P}} \models \psi$ iff $V^{\mathbb{P}*\dot{\mathbb{Q}}} \models \psi$. For example, it is a theorem of Woodin that if there is a proper class of cardinals which are either measurable Woodin or strongly compact, then generic invariance of Σ^2_1 holds with respect to CH (see for example [33, Theorem 3.2.1]).

[11]It is not accurate to express it in a Σ^2_2 way, even though the relation ψ is Δ^2_2 in $V[G_{\mathbb{Q}}][g]$: Let ψ_1 and ψ_2 be Σ^2_2-formulas such that for all reals r, s, $\psi(r, s) \Leftrightarrow \psi_1(r, s) \Leftrightarrow \neg\psi_2(r, s)$.

The fact that ψ defines a well-ordering of $\mathbb{R}$ can be formalized as follows:

$$\forall x, y, z \in \mathbb{R}\, \big[(\psi(x, y) \vee x = y \vee \psi(y, x)) \wedge (\neg\psi(x, x)) \wedge (\psi(x, y) \to \neg\psi(y, x)) \\ \wedge (\psi(x, y) \wedge \psi(y, z) \to \psi(x, z)) \wedge \exists n\, \neg\psi(x_{n+1}, x_n)\big],$$

where $x \mapsto \langle x_n : n < \omega \rangle$ is some recursive bijection between $\mathbb{R}$ and $\mathbb{R}^\omega$. This statement can certainly be expressed in a Σ^2_2 way if ψ_1 and ψ_2 are judiciously used in place of ψ in the displayed formula above. However, we *must* add to it the clause that $\psi_1(x, y) \leftrightarrow \neg\psi_2(x, y)$ (which is not a Σ^2_2 statement), since this equivalence is certainly vital for the validity of the assertion that ψ defines a well-ordering, but it is not a theorem and must therefore be explicitly claimed.

4. A result of Kunen

In this section we sketch a proof (which is probably folklore) of the result of Kunen mentioned in Remark 3.6. We use the technique illustrated in the previous section of finding, in the ground model, filters that are generic over inner models for forcing notions that are sufficiently closed.

Theorem 4.1 (Kunen). *Assuming the consistency of measurable cardinals, it is consistent that there are models $M \subset W \subset V$ and a cardinal κ such that κ is measurable in M and V but not in W.*

Kunen's argument is different from the one to follow. He starts with a ground model N where there is a measurable cardinal κ, and adds a generic S to N for the *Silver preparation forcing* so, in $M = N[S]$, κ is measurable, and it remains measurable after adding to M a Cohen subset of κ. In $W = M[G]$ there is a κ-Suslin tree (so κ is not measurable). G is generic for a forcing like the Prikry-Silver forcing ([9, §7]), but care is taken to ensure that the tree that is added is homogeneous. Finally, in $V = W[H]$ the tree is killed in the usual way. The iterated forcing that first adds G and then adds H is equivalent to adding over M a Cohen subset of κ, so κ is measurable in V. The details of this argument can be found in [30].

In the argument below, we avoid the need for the preparation forcing by working over a nice inner model. Recall:

Definition 4.2. By $L[\mu]$ we mean the smallest proper class inner model of the theory

$$\mathsf{ZFC} + \text{“There exists a measurable,”}$$

in this context, by μ we always mean a witness to measurability, i.e.,

$$L[\mu] \models \mu \text{ is a normal } \kappa\text{-complete measure on some cardinal } \kappa,$$

and by smallest we mean that κ is as small as possible ($L[\mu]$ is sometimes called the *core ρ-model*, see [15, Definition 13.8]).

We abuse notation in the usual way, and occasionally talk about the theory $V = L[\mu]$. The minimality assumption is just adopted for definiteness, and not required in the arguments; of course if, for some U and λ,

$$L[U] \models U \text{ is a normal } \lambda\text{-complete measure on } \lambda,$$

then $L[U] \models V = L[\mu]$.

Proof. The idea of the proof is to start with a measurable cardinal κ and a stationary set $S \subset \kappa^+$. We carefully associate to S a sequence

$$\langle S_\delta : \delta < \kappa \text{ inaccessible} \rangle,$$

$S_\delta \subset \delta^+$ stationary, in such a way that the stationarity of the S_δ can be destroyed by forcing while the stationarity of S is preserved. By a reflection argument, this will contradict the measurability of κ in the extension, but a further extension (destroying the stationarity of S) resurrects its measurability, providing the example we desire.

Recall ([1, Definition 1.1]) that a stationary subset S of a regular cardinal λ is called *fat* iff for every club $C \subseteq \kappa$, $S \cap C$ contains closed sets of ordinals of arbitrarily large order types below κ. It is shown in [1, Theorem 1] that if $\lambda = \rho^+$ where $\rho^{<\rho} = \rho$, $2^\rho = \lambda$, and $S \subseteq \lambda$ is fat, then there is a λ-distributive forcing $\mathbb{P} = \mathbb{P}_S$ such that $|\mathbb{P}| = \lambda$ and $\mathbb{P}$ adds a club $C \subseteq S$. $\mathbb{P}$ is just the set of bounded, closed subsets of S, ordered by end extension. It follows from [1, Lemma 1.2] that if $\lambda = \rho^+$ where ρ is regular, $A \subset \{\, \alpha < \lambda : \operatorname{cf}(\alpha) = \rho \,\}$, and $\{\, \alpha \in \lambda \setminus A : \operatorname{cf}(\alpha) = \rho \,\}$ is stationary, then $\lambda \setminus A$ is fat. In this case, if GCH holds, then $\mathbb{P}_{\lambda \setminus A}$ is η-closed for all $\eta < \rho$.

Work in $L[\mu]$. Let $j_\mu : L[\mu] \to M_1$ be the embedding by the normal measure μ, and let $\kappa = \operatorname{cp}(j_\mu)$, so $\mathcal{P}(\kappa)^{L[\mu]} = \mathcal{P}(\kappa)^{M_1}$ and, in particular, $(\kappa^+)^{L[\mu]} = (\kappa^+)^{M_1}$.

Notice that there is a set $S \in M_1$ such that S is a stationary subset of κ^+ in $L[\mu]$, $\forall \alpha \in S\, (\operatorname{cf}(\alpha) = \kappa)$, and $\{\, \alpha \in \kappa^+ \setminus S : \operatorname{cf}(\alpha) = \kappa \,\}$ is also stationary in $L[\mu]$ (for example, because a $\kappa \times \kappa^+$ Ulam matrix defined in M_1 would still be an Ulam matrix in $L[\mu]$, see [31, Theorem II.6.11]).

Fix a function f such that $j_\mu(f)(\kappa) = S$, so for μ-almost every inaccessible $\delta < \kappa$, $f(\delta)$ is a stationary subset of δ^+ concentrating on the ordinals of cofinality δ such that $\{\, \alpha \in \delta^+ \setminus f(\delta) : \operatorname{cf}(\alpha) = \delta \,\}$ is also stationary. By redefining f on a μ-measure zero set, if necessary, we may assume that this holds for all inaccessible cardinals $\delta < \kappa$.

Consider the Backward Easton support iteration $\mathbb{P}_\kappa$ of forcings $\mathbb{F}_\delta$, $\delta < \kappa$ inaccessible, such that $\mathbb{F}_\delta = \mathbb{P}_{\delta^+ \setminus f(\delta)}$ adds a club subset of $\delta^+ \setminus f(\delta)$. Notice that $\mathbb{P}_\kappa$ is κ-cc ([9, Corollary 2.4]) and that, for every inaccessible δ, $\mathbb{P}_\kappa$ factors as $\mathbb{Q}_\delta * \mathbb{F}_\delta * \mathbb{Q}^\delta$ where $\mathbb{Q}^\delta$ is, say, δ^{++}-closed, and $\mathbb{Q}_\delta$ preserves the stationarity of $f(\delta)$.

By elementarity, in M_1, $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{F}_\kappa * \mathbb{Q}^\kappa$, where $\mathbb{F}_\kappa$ adds a club subset of $\kappa^+ \setminus S$ and $\mathbb{Q}^\kappa$ is κ^{++}-closed.

Let G_κ be $\mathbb{P}_\kappa$-generic over $L[\mu]$, and let g be $\mathbb{F}_\kappa$-generic over $L[\mu][G_\kappa]$.

Claim 4.3. *κ is measurable in $L[\mu][G_\kappa][g]$.*

Proof. We find in $L[\mu][G_\kappa][g]$ a lifting

$$j : L[\mu][G_\kappa] \xrightarrow{\prec} M_1[j(G_\kappa)]$$

of j_μ. This suffices because $\mathcal{P}(\kappa)^{L[\mu][G_\kappa]} = \mathcal{P}(\kappa)^{L[\mu][G_\kappa][g]}$, so the $L[\mu][G_\kappa]$-ultrafilter derived from j is an ultrafilter on κ, and it is straightforward to verify that it is non-principal and κ-complete.

The lifting is found arguing as in Claim 3.2: $\mathbb{Q}^\kappa$ is at least κ^+-closed (i.e., closed under extensions of decreasing sequences of length $< \kappa^+$) in $M_1[G_\kappa][g]$ and, in fact, it is κ^+-closed in $L[\mu][G_\kappa][g]$. This follows from standard arguments about Backward Easton iterations, and is almost verbatim as [14, Lemma 11.3 and Lemma 11.6], to which we refer for further details. Since, in $L[\mu][G_\kappa][g]$, $|j(\kappa)| =$

κ^+, there is in $L[\mu][G_\kappa][g]$ an $M_1[G_\kappa][g]$-generic filter H. Defining $j(G_\kappa) = G_\kappa * g * H$, j_μ lifts in the usual way to an embedding j as required.

This concludes the proof of Claim 4.3. □

Claim 4.4. *κ is not measurable in $L[\mu][G_\kappa]$.*

Proof. Suppose otherwise, and let $k : L[\mu][G_\kappa] \to N$ be the corresponding embedding coming from a normal measure on κ. Then $k\restriction(\mathcal{P}(\kappa) \cap L[\mu])$ is the restriction to $\mathcal{P}(\kappa) \cap L[\mu]$ of an iteration of j_μ (see for example [26, Exercise 20.13]), and therefore $k(f)(\kappa) = j_\mu(f)(\kappa) = S$.

But then, by elementarity, $k(G_\kappa)$ adds a club set killing the stationarity of S. This contradicts that $\mathbb{P}_\kappa$ is κ-cc. □

Theorem 4.1 follows at once, taking $M = L[\mu]$, $W = L[\mu][G_\kappa]$, and $V = L[\mu][G_\kappa][g]$. □

Remark 4.5. The use of $L[\mu]$ in the previous argument is by no means essential: An additional preparation forcing (ensuring that in any future extension, for any embedding k with critical point κ, $k(f)(\kappa) = S$) would allow us to start with an arbitrary ground model (of GCH) instead of $L[\mu]$. See [22, Theorem 2] for an elaboration on the argument above that produces an example where κ is atomlessly measurable in the final model V, but the reals of V are not obtained by adding random reals to any inner model where κ is measurable.

5. Anticoding results

The results in this section are folklore, although our presentation may be novel. Of course, the complexity Δ^2_2 of the well-ordering we obtained in section 3 is an overkill; notice the third-order universal quantifier only ranges over *bounded* subsets of κ. It is natural to wonder whether we can improve the complexity of the well-ordering to be Σ^2_1. The problem with following a strategy similar to the one described in 3 is that we need to ensure correctness of the model M with respect to the combinatorial structure of the universe that carries out the coding (the club base numbers, for example). This level of correctness needs to be attained via projective and (at most) third-order existential statements. This seems to suggest that we need to be able to code (suitable) bounded subsets of κ by reals. In general (as in the arguments of [3] and [4]), this is done by arranging that the universe satisfies something like a sufficiently strong fragment of MA to be able to use the coding provided by almost-disjoint forcing.

Unfortunately (as it is well known, see [37]) MA itself fails after adding even one random real, so it is incompatible with real-valued measurability of the continuum. For example:

Theorem 5.1. *If $\mathsf{RVM}(\kappa)$ holds and $\kappa \le \mathfrak{c}$, then there is a ccc partial order $\mathbb{P}$ such that $\mathbb{P} \times \mathbb{P}$ is not ccc.*

Corollary 5.2. *If $\mathsf{RVM}(\kappa)$ holds and $\kappa \le \mathfrak{c}$, then MA_{ω_1} fails.* □

The hypothesis we display is not ideal, but there is some subtlety here, since Prikry showed that MA is compatible with quasi-measurability of the continuum, see [19, Proposition 9G].

Remark 5.3. Corollary 5.2 has been shown in many ways independently of Theorem 5.1. Arguments more in the spirit of forcing axioms are possible: For example, if κ is atomlessly measurable, then

- $\mathrm{non}(\mathbb{R}, \mathcal{N}) = \mathrm{cov}(\mathbb{R}, \mathcal{M}) = \mathrm{add}(\mathcal{M}) = \mathrm{add}(\mathcal{N}) = \mathfrak{p} = \omega_1$. Here, $\mathcal{N}$ is the ideal of Lebesgue null sets and $\mathcal{M}$ is the ideal of meager sets.
- $\mathfrak{b} < \kappa$.

See [19] and references within. The particular case $\mathsf{RVM}(\mathfrak{c}) \Rightarrow \mathfrak{b} < \mathfrak{c}$ is due to Banach-Kuratowski [7].

It is a well-known result (due to Bell) that $\mathfrak{p}$ is the smallest cardinal λ such that MA_λ(σ-centered) fails, see [17, §14]. Recall that almost disjoint forcing is σ-centered.

Proof of Theorem 5.1. This is a corollary of the following result of Roitman[12] [37]:

Lemma 5.4. *In $V^{\mathtt{Random}_\omega}$ there is a ccc partial order whose square is not ccc.* □

Corollary 5.5. *Roitman's result* 5.4 *holds in $V^{\mathtt{Random}_\lambda}$ and not just $V^{\mathtt{Random}_\omega}$.*

Proof. Since $\mathtt{Random}_\lambda/\mathtt{Random}_\omega \cong \mathtt{Random}_\lambda$, Corollary 5.5 follows from Corollary 1.32. □

Assume $\mathsf{RVM}(\kappa)$ where $\kappa \leq \mathfrak{c}$, and let λ be such that in $V^{\mathtt{Random}_\lambda}$ there is an embedding $j : V \to N$ with $\mathrm{cp}(j) = \kappa$. By Corollary 5.5 there is in $V^{\mathtt{Random}_\lambda}$ a ccc partial order $\mathbb{P}$ whose square is not ccc. By taking Skolem hulls, we may assume $|\mathbb{P}| \leq \aleph_1$. By Remark 1.21, we may as well assume N is closed under ω_1-sequences, so we can take $\mathbb{P} \in N$ and

$$N \models \mathbb{P} \text{ is ccc but } \mathbb{P} \times \mathbb{P} \text{ is not.}$$

But then, by elementarity, there is such a partial order in V. □

Question 5.6 (Fremlin). *Suppose κ is atomlessly measurable. Are there two ccc posets $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ has an antichain of size κ?*

Another forcing axiom that is used to code information about subsets of reals is the Open Coloring Axiom OCA, see [20].

Theorem 5.7. *If* $\mathsf{RVM}(\kappa)$ *holds and* $\kappa \leq \mathfrak{c}$*, then* OCA *fails.*

This is essentially due to Todorčević.

Proof. The key to this result is the notion of an entangled linear order, see [41]. The following is [41, Theorem 2]:

Lemma 5.8. *If E is a set of random reals, then E is ω_1-entangled.* □

[12] In [8, Theorem 3.2.30], this is erroneously attributed to Galvin. Galvin devised a general method to construct such posets. Roitman showed that the construction works in $V^{\mathbb{F}}$, where $\mathbb{F} = \mathtt{Add}(\omega, 1)$ or $\mathbb{F} = \mathtt{Random}_\omega$.

I think the following is due to Todorčević and Baumgartner, see [20] for a proof.

Fact 5.9. *If there is an uncountable ω_1-entangled subset of $\mathbb{R}$, then* OCA *fails*[13]. □

Theorem 5.7 now follows as before: For some λ, in $V^{\mathtt{Random}_\lambda}$ there is an embedding $j : V \to N$ with $\mathrm{cp}(j) = \kappa$ and ${}^{\omega_1}N \subseteq N$, so in N there is an uncountable ω_1-entangled subset of $\mathbb{R}$ and, by elementarity, there is such a set also in V. By Fact 5.9, OCA fails in V.[14] □

These arguments should make it clear that any statement sufficiently fragile in the sense that random forcing destroys it and sufficiently absolute in the sense that it transfers to the generic ultrapower of the ground model, is bound to fail if there are atomlessly measurable cardinals. Thus, any naive attempt to improve the complexity of the well-ordering obtained in Section 3 by coding bounded subsets of κ by reals (where κ was measurable in the ground model and turns atomlessly measurable in the extension), say by including into the product we were calling $\mathbb{P}$ small factors that will do the coding of bounded subsets, runs into the immediate difficulty that we are adding random reals by homogeneous forcing (by the poset we were calling $\mathbb{Q}$, which is just $\mathtt{Random}_\kappa$), which most likely will undo our coding. We would then have to do the coding in such a way that no initial segment of the iteration would suffice to code a bounded set of κ in the final model, but this seems difficult as well, because bounded sets of κ would most likely appear in initial segments of the iteration.

This section has highlighted inherent difficulties that a proof of the consistency of $\mathsf{RVM}(\mathfrak{c})$ together with a Σ^2_1-well-ordering of the reals must face.

Woodin's result in the following section solves them in an indirect manner, by restricting in a very serious way the universe over which the argument takes place. The question of whether measurability of κ and GCH (or for that matter, any set of hypotheses which do not carry anti-large cardinal restrictions, or smallness requirements on the universe) suffice to force a model of $\mathsf{RVM}(\mathfrak{c})$ with a Σ^2_1-well-ordering of $\mathbb{R}$ is still open. In section 7 we discuss an alternative approach.

6. Σ^2_1-well-orderings

The result of this section is due to Woodin.

Assume that $V \models \kappa$ is measurable and $2^\kappa = \kappa^+$, and let $j : V \to N$ be a normal ultrapower embedding with $\mathrm{cp}(j) = \kappa$.

Let $\mathbb{Q} = \mathtt{Random}_\kappa$ and $\mathbb{P}$ be the Easton product over the inaccessible cardinals $\lambda < \kappa$ of $\mathtt{Add}(\lambda^+, 1) \times \mathtt{Add}(\lambda^{++}, 1)$.

Force over V with $\mathbb{P} \times \mathbb{Q}$, and let $G_\mathbb{P} \times G_\mathbb{Q}$ be generic.

[13]The existence of uncountable ω_1-entangled subsets of $\mathbb{R}$ also contradicts MA_{ω_1} (see [41]), thus giving yet another proof of Corollary 5.2.

[14]This argument actually shows that if κ is atomlessly measurable, then for every $\lambda < \kappa$ there is an ω_1-entangled subset of $\mathbb{R}$ of size λ.

As before:

- If $j(\mathbb{P}) = \mathbb{P} \times \mathbb{P}_{\text{tail}}$, then there is $G^* \in V$, $\mathbb{P}_{\text{tail}}$-generic over N, such that j lifts to $j_1 : V[G_{\mathbb{P}}] \to N[G_{\mathbb{P}}][G^*]$.
- $j(\mathbb{Q})/\mathbb{Q}$ is isomorphic to an appropriate random forcing in any intermediate model between $V[G_{\mathbb{Q}}]$ and $V_1 := V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$, inclusive, and $\mathfrak{c} = \kappa$ is real-valued measurable in V_1. In fact if H is $j(\mathbb{Q})/\mathbb{Q}$-generic over V_1 then, in $V_1[H]$, j lifts to $j_2 : V_1 \to N[G_{\mathbb{P}}][G^*][G_{\mathbb{Q}}][H]$, thus showing $\mathsf{RVM}(\mathfrak{c})$ in V_1, by Solovay's Theorem 1.6.
- Similarly, in $V[G_{\mathbb{Q}}][H]$, j lifts to $j_3 : V[G_{\mathbb{Q}}] \to N[G_{\mathbb{Q}}][H]$.
- $\mathbb{R}^{V[G_{\mathbb{Q}}]} = \mathbb{R}^{V[G_{\mathbb{Q}}][G_{\mathbb{P}}]}$.

In $V[G_{\mathbb{Q}}]$, let $A \subset \kappa$ code a well-ordering of $\mathbb{R}$ in order type κ.

Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ be the increasing enumeration of the inaccessible cardinals in V below κ. For $\alpha < \kappa$, let G_α be the α^{th} component of $G_{\mathbb{P}}$, so G_α is the product of an $\mathtt{Add}(\delta_\alpha^+, 1)$-generic and an $\mathtt{Add}(\delta_\alpha^{++}, 1)$-generic over V. Let G^*_α be the $\mathtt{Add}(\delta_\alpha^+, 1)$-generic, if $\alpha \in A$, and the $\mathtt{Add}(\delta_\alpha^{++}, 1)$-generic, if $\alpha \notin A$. Finally, let

$$g = \prod_{\alpha<\kappa} G^*_\alpha.$$

Notice A is definable from g.

The same argument as in Claim 3.5 shows G^* and $j_3(A)$ suffice to define $j_2(g)$ (and recall $G^* \in V$ and $j_3(A) \in V[G_{\mathbb{Q}}][H]$). It follows as in that claim that $\mathfrak{c} = \kappa$ is real-valued measurable in $V[G_{\mathbb{Q}}][g]$, and that a lifting of j to $j^* : V[G_{\mathbb{Q}}][g] \to N[G_{\mathbb{Q}}][H][j_2(g)]$ definable in $V[G_{\mathbb{Q}}][g][H]$ serves as a witness.

Theorem 6.1 (Woodin). *If $V = L[\mu]$, then in $V[G_{\mathbb{Q}}][g]$, $\mathsf{RVM}(\mathfrak{c})$ and there is a Δ^2_1-well-ordering of $\mathbb{R}$.*

Proof. Let $V_1 = L[\mu][G_{\mathbb{Q}}][g]$, so $\mathbb{R}^{V_1} = \mathbb{R}^{L[\mu][G_{\mathbb{Q}}]}$ and $\mathsf{RVM}(\mathfrak{c})$ holds in V_1. We claim that the well-ordering coded by A is Σ^2_1 in V_1. This we verify by "guessing" the ground model. What the following claim formalizes is our intuition that any structure which resembles $L[\mu]$ sufficiently close must coincide with $L[\mu]$. This resemblance we indicate in terms of a covering property.

Definition 6.2. Let N be a transitive structure that models enough set theory. We say that N *satisfies countable covering* iff

$$\forall \sigma \in \mathcal{P}_{\omega_1}(N)\, \exists \tau \in N\, (\sigma \subseteq \tau \text{ and } N \models |\tau| \leq \aleph_0).$$

Once again, we use $\mathsf{ZFC}^{-\varepsilon}$ to denote a sufficiently strong fragment of ZFC, say (as before), $\mathsf{ZFC}{\restriction}\Sigma_{200}$, ZFC with the replacement schema restricted to Σ_{200} statements. Obviously, much less suffices.

Claim 6.3. *In V_1, suppose M is transitive, $|M| = \mathfrak{c}$, $M \models \mathsf{ZFC}^{-\varepsilon} + V = L[\mu]$. Let κ_M be the measurable cardinal in the sense of M, and $\kappa = \mathfrak{c}$. Suppose $\kappa_M \geq \mathfrak{c}$, M is iterable and satisfies countable covering. Then $M_\kappa = L[\mu]_\kappa$.*[15]

The hypothesis of Claim 6.3 requires some expansion. The point of the claim is that we have identified the ground model (or, better, the part of the ground model relevant to our argument) in a projective fashion.

See [15] for a careful exposition of iterability at this level and for the necessary background on the argument to follow. What we refer to as K^{DJ} is just called K in [15], and $L[\mu]$ is called there $L[U]$. K^{DJ} is the Dodd-Jensen core model.

Proof. First notice that an initial segment of $L[\mu]$ itself satisfies the requirements: Iterability is clear, and countable covering holds because $\mathbb{Q}$ is ccc and $\mathbb{P}$ is ω_1-closed.

Assume M satisfies the requirements of the claim. Notice that (provably in ZFC + "$L[\mu]$ exists"), $K^{DJ}_\kappa = L[\mu]_\kappa$. It follows that $M_\kappa \models \mathsf{ZFC} + V = K^{DJ}$ (ZFC holds in M_κ because GCH holds in M, so $M \models \kappa$ is strongly inaccessible). So $M_\kappa = (K^{DJ})^{M_\kappa} = K^{DJ} \cap M_\kappa \subseteq K^{DJ} \cap V_\kappa = L[\mu]_\kappa$, where we use [15, Lemma 14.18] to justify the equality $(K^{DJ})^{M_\kappa} = K^{DJ} \cap M_\kappa$ (namely, K^{DJ} is the union of all the mice in the sense of [15] if $0^\sharp$ exists, but being a mouse relativizes downwards).

If $M_\kappa \subsetneq K^{DJ}_\kappa$, then there is a least *sharplike* mouse $\bar{M} \notin M_\kappa$ such that $M_\kappa \subsetneq L[\bar{M}]_\kappa$ (see [15, Chapter 15]). Notice that $K^{DJ}_\kappa \models L[\mu]$ does not exist, because κ is a cardinal in V_1 (otherwise, in V_1, $L\big[L[\mu]^{K^{DJ}_\kappa}\big]$ would really be a model $L[U]$ with U a normal measure in $L[U]$ on some cardinal $\lambda < \kappa$, contradicting the minimality of κ in V_1. It follows from [15, Chapter 16] that there is a nontrivial $j : M_\kappa \xrightarrow{\prec} M_\kappa$ in $L[\bar{M}]_\kappa$, and this certainly contradicts the countable covering property of M considering, for example, the first ω terms of the critical sequence derived from j.

This completes the proof of Claim 6.3. □

We are basically done now: To require iterability of a model M as in Claim 6.3 is a projective requirement; for example, if $M \models V = K^{DJ}$, iterability of M states that every countable premouse (that we can code with a real) that embeds into M in a Σ_1-elementary way is iterable (if M is coded by a set of reals, the existence of this embedding is an assertion about an ω-sequence of reals, coding the range of the embedding, and about the satisfaction relation between a universal Σ_1 formula and the elements of M; all of this can be expressed in a projective fashion; see [15, Lemma 8.7] for a proof of the claimed characterization of iterability). The iterability of a countable premouse is in turn a Π^1_2 statement (uniformly in a real

[15] The notation we use here is ambiguous. For N a model and α an ordinal,

$$N_\alpha = \{\, x \in N : \mathrm{rk}\,(x) < \alpha \,\},$$

where $\mathrm{rk}\,(x)$ is the set-theoretic rank of x. In particular, $L[\mu]_\kappa$ is *not* the κ^{th}-stage in the (classical) constructible hierarchy of $L[\mu]$.

coding the premouse as a parameter), see [15, Lemma 13.21]. Hence, to define A in a Σ^2_1-way following the approach explained at the end of section 2 it suffices to notice the following claim, whose proof concludes the proof of Theorem 6.1.

Claim 6.4. *In V_1 suppose $\hat\delta < \kappa$ and $a \subseteq \hat\delta^+$ is such that $a \notin L[\mu]$ is $\mathtt{Add}(\hat\delta^+, 1)$-generic over $L[\mu]$. Then $\hat\delta$ is an inaccessible cardinal δ_β or its successor, and $\beta \in A$ iff $\hat\delta = \delta_\beta$.*

It follows that A can be defined by refering to those cardinals $\hat\delta$ for which there is a set a as above.

Proof. This follows quite easily by what is essentially the decoding argument given during the proof of Theorem 3.1. □

This completes the proof of Theorem 6.1. □

Notice that essentially the same argument provides models of a Σ^2_1-well-ordering together with $\mathsf{RVM}(\mathfrak{c})$, as long as the ground model is fine structural, and the iterability condition for countable mice is projective[16].

Following this approach, granting large cardinals, and starting with a definable fine structural model, the construction produces a model of $\mathsf{RVM}(\mathfrak{c})$ together with a $\Sigma^2_1(\mathrm{Hom}_\infty)$-well-ordering of $\mathbb{R}$. Here, $\Sigma^2_1(\mathrm{Hom}_\infty)$ is the pointclass of sets of reals A such that for some projective formula ψ and some real parameter r, A can be defined by: For all $s \in \mathbb{R}$,

$$s \in A \iff \exists B\, (\psi(s, r, B) \text{ and } B \in \mathrm{Hom}_\infty).$$

The pointclass Hom_∞ consists of all ∞-Homogeneous sets of reals. Under the background assumption that there are unboundedly many Woodin cardinals (in V, not necessarily in the fine structural model), it coincides with the pointclass of all Universally Baire sets of reals. See [39] for definitions, details and references.

7. Real-valued measurability and the Ω-conjecture

This section announces an improvement due to Woodin of the result in section 6. We include enough definitions to make the statement meaningful.

Recall we have shown inherent difficulties to a straightforward attempt to obtain (without anti-large cardinal assumptions) extensions of the universe where $\mathfrak{c}$ is real-valued measurable and there are Δ^2_1-well-orderings of $\mathbb{R}$. The specific technical difficulty that must be resolved is whether it is possible to devise a coding of bounded subsets of $\mathfrak{c}$ by reals. The usual way of obtaining such coding

[16]If M_κ is the model the corresponding version of Claim 6.3 tries to identify, a fake candidate would give rise to a club of inaccessible cardinals below the distinguished measurable κ, again violating covering. Recall that iterability of a fine structural premouse M is in essence a condition about its countable elementary substructures, and that, in the presence of only finitely many Woodin cardinals, this condition is a projective requirement. See for example the introduction to [36].

is by ensuring that some kind of forcing axiom holds. However, we have shown that real-valued measurability contradicts even very general schema toward such forcing axioms. The way this difficulty was dealt with in the previous section was by circumventing it, by working within a "thin" ground model which could therefore be identified in a projective fashion in the relevant forcing extension.

Woodin's idea is to exploit this "thinness" within a broader context. Specifically, instead of trying to establish directly that a Δ^2_1-well-ordering of $\mathbb{R}$ and $\mathsf{RVM}(\mathfrak{c})$ can be added by forcing, he settles for showing the Ω-consistency of this assumption. We proceed now to present a brief summary of Ω-logic, of Ω-consistency, of its connection with the problem of showing consistency via forcing, and close with the statement of Woodin's result and the question of possible generalizations. The reader may also want to look at [5], in this same volume, where Ω-logic is studied in some detail and its basic theorems are established.

In [44], Woodin introduces Ω-logic as a strong logic extending first-order logic (in fact, extending β-logic), and uses it to argue for a negative solution to Cantor's continuum problem. His argument would justify the adoption of $\neg\mathsf{CH}$ if a particular conjecture holds. This conjecture would show that Ω-logic is in a sense as strong as possible for a wide class of statements (including CH). We advise the interested reader to consult [44] for more details. All the results and definitions presented here, unless otherwise explicitly stated, are due to Woodin. However, it must be pointed out that since the appearance of [44] and even [45], the basic definitions have changed somewhat, see [46]. In particular, the definition of Ω-logic we state below is purely semantic, and corresponds to what [45] calls Ω^*-logic. This move requires a slight change in the definition of *proofs* in Ω-logic, as we will explain.

The concept of strong logic is defined in [45]. We do not need it here, but it is useful to mention that we are only interested in it with respect to theories (in a first-order language) extending ZFC. Ω-logic and first-order logic are both examples of strong logics, at opposite ends of the spectrum, first-order logic being the most generous strong logic there is, in the sense that it allows as many structures as possible, and we regard this generosity as a weakness. On the other hand, Ω-logic is the strongest possible logic, allowing only those structures that pass for acceptable models of set theory, under reasonable requirements of acceptability. For example, while first-order logic allows any structure of the form (M, E) as a possible model, ω-logic only allows those structures that "compute V_ω correctly" and β-logic only allows those structures that are correct about well-foundedness. Ω-logic goes as far in this direction as possible, subject to natural requirements that we list below.

Recall that if M is a transitive structure, $M_\alpha = \{\, x \in M : \mathrm{rk}\,(x) < \alpha \,\}$, see also [5, §1.1]. The following is also [5, Definition 1.7].

Definition 7.1 (Ω-logic). Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then

$$T \models_\Omega \phi$$

iff for all $\mathbb{P}$ and all λ, if $V^{\mathbb{P}}_\lambda \models T$, then $V^{\mathbb{P}}_\lambda \models \phi$.

Remark 7.2. According to this definition, an Ω-*satisfiable* sentence ϕ, i.e., a sentence ϕ such that $\neg\phi$ is not Ω-valid, is one such that for some $\mathbb{P}$ and α, $V_\alpha^{\mathbb{P}} \models \mathsf{ZFC} + \phi$, see [5, Definition 3.1]. It is easy to see that if ϕ is Σ_2 and Ω-satisfiable, then in fact ϕ is forceable *over* V, i.e., for some $\mathbb{P}$, $V^{\mathbb{P}} \models \phi$. In effect, let $\phi \equiv \exists x\, \psi(x)$ be a Σ_2 sentence, where $\psi(x)$ is Π_1. Suppose ϕ is Ω-satisfiable, and let $\alpha, \mathbb{P}$ be such that $V_\alpha^{\mathbb{P}} \models \mathsf{ZFC}+\phi$. Let u be such that $V_\alpha^{\mathbb{P}} \models \psi(u)$ and let $\omega < \kappa < \alpha$ be a cardinal in $V_\alpha^{\mathbb{P}}$ (and therefore in $V^{\mathbb{P}}$), sufficiently large so $u \in H_\kappa^{\mathbb{P}}$. A well-known result of Levy (see [35]) asserts that whenever $\lambda > \omega$ is a cardinal, $H_\lambda \prec_1 V$. Relativizing Levy's result to $V_\alpha^{\mathbb{P}}$, it follows that $H_\kappa^{\mathbb{P}} \models \psi(u)$. Applying Levy's result in $V^{\mathbb{P}}$, we see that $V^{\mathbb{P}} \models \phi$, as wanted.

A logic (in the sense of a satisfaction relation between first-order structures and first-order statements) satisfying the definition of Ω-logic (and, perhaps, being more restrictive) is said to be *generically sound*.

An important difference between first-order logic and Ω-logic is that the latter requires a healthy large cardinal structure on the background universe for certain absoluteness requirements to hold; this absoluteness is essential for a reasonable study of Ω-logic. For this section, let us define:

Definition 7.3. By *our Base Theory* we mean

$$\mathsf{ZFC} + \text{“There is a proper class of Woodin cardinals.”}$$

The following is proven in [5, Theorem 1.8].

Theorem 7.4 (Generic Invariance). *Assume our Base Theory. Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then $T \models_\Omega \phi$ iff for all $\mathbb{P}$, $V^{\mathbb{P}} \models T \models_\Omega \phi$.* □

Corresponding to the semantic notion of satisfiability we want to develop a syntactic counterpart, $\vdash_\Omega$. Recall that proofs in first-order logic can be construed as certain trees. Similarly, for Ω-logic, we develop a notion of *certificate* that plays this role.

The certificates in this case are more specialized, and it is better to present first the sets in terms of which we are to define them, the Universally Baire sets, which we introduce directly in the way we need them, by what is usually stated as a corollary of their standard definition. See also [5, §2.1].

Definition 7.5 (Feng, Magidor, Woodin [16]**).** Let λ be an infinite cardinal. A set $A \subseteq \omega^\omega$ is λ-*Universally Baire* iff there are λ-*absolutely complementing* trees for A, i.e., a pair T, T^* of trees on $\omega \times X$ for some X, such that

1. $A = p[T]$ and $\omega^\omega \setminus A = p[T^*]$.
2. $\mathbb{1} \Vdash_{\mathbb{P}} p[T] \cup p[T^*] = \omega^\omega$ for any forcing $\mathbb{P}$ of size at most λ.

A is ∞-Universally Baire or, simply, Universally Baire, iff it is λ-Universally Baire for all λ.

Notice that if A is λ-Universally Baire, and $T, T^*, \mathbb{P}$ are as above, then $\mathbb{1} \Vdash_{\mathbb{P}} p[T] \cap p[T^*] = \emptyset$.

The Universally Baire sets generalize the Borel sets and have all the usual regularity properties.

Under reasonable large cardinal assumptions, the pointclass of Universally Baire sets is quite closed. For example (see [16] for the case $A = \mathbb{R}$ or [34] for the general case):

Fact 7.6. *Assume our Base Theory. Suppose A is Universally Baire. Then every set of reals in $L(A, \mathbb{R})$ is Universally Baire.* □

There are somewhat cleaner ways of stating this fact. For example, since our Base Theory grants that every set has a sharp, Fact 7.6 is equivalent to (see [34]):

Fact 7.7. *Assume our Base Theory. Suppose A is Universally Baire. Then $A^\sharp$ is Universally Baire.* □

Given such a set A, it makes sense to talk about its *interpretation* in extensions of the universe, in what generalizes the idea of Borel codes for Borel sets.

Definition 7.8. Let A be Universally Baire. Let $\mathbb{P}$ be a forcing notion, and let G be $\mathbb{P}$-generic over V. Then the interpretation A_G of A in $V[G]$ is

$$A_G = \bigcup \{ p[T] : T \in V \text{ and } V \models A = p[T] \}.$$

This is the natural notion we would expect: If T, T^* are λ-complementing trees such that $p[T] = A$, if $|\mathbb{P}| \leq \lambda$ and G is $\mathbb{P}$-generic over V, then $V[G] \models A_G = p[T]$.[17]

The certificates for Ω-logic are issued in terms of Universally Baire sets, and thus we arrive at the concept of A-closed structures. See also [5, §2.2].

Definition 7.9. Let $A \subseteq \omega^\omega$ be Universally Baire. A transitive set M is *A-closed* iff for all $\mathbb{P} \in M$ and all $\mathbb{P}$-terms $\tau \in M$,

$$\{ p \in \mathbb{P} : V \models p \Vdash \tau \in A_G \} \in M.$$

Remark 7.10. In practice, countable transitive A-closed models M are those admitting a pair of "absolutely complementing with respect to M" trees $T, T^* \in M$ such that the interpretation of A (which needs not be in M) would be in forcing extensions of M by forcing notions in M given by the projection of T, and such that in V, $p[T] \subseteq A$ and $p[T^*] \subseteq \mathbb{R} \setminus A$. Notice that M-generics for forcing notions in M exist in V, since M is countable.

Even though the official definition restricts the A-closed structures from the beginning to transitive sets, it may be helpful to point out that β-logic can be characterized in terms of A-closure: An ω-model $(M, E) \models \mathsf{ZFC}$ is well founded iff,

[17] A word of warning is in order: Suppose $A = \{ r \in \mathbb{R} : \varphi(r) \}$ is Universally Baire, where φ is, say, Σ^1_3. It does *not* follow that

$$A_G = \{ r \in \mathbb{R}^{V[G]} : \varphi(r) \}. \qquad (*)$$

Universally Baire sets figure prominently in generic absoluteness arguments but, in addition, equalities like $(*)$ need to be ensured. See for example [39].

under the proper interpretation, it is A-closed for each Π^1_1-set A, see [5, Theorem 2.23] for a proof.

The following is [44, Lemma 10.143], see [5, Proposition 2.9] for a proof.

Theorem 7.11 (Woodin). *Let $M \models \mathsf{ZFC}$ be transitive, and let A be Universally Baire. Then the following are equivalent:*

1. *M is A-closed.*
2. *Suppose $\mathbb{P} \in M$ and G is $\mathbb{P}$-generic over V. Then*

$$V[G] \models A_G \cap M[G] \in M[G]. \qquad \square$$

With the concept of A-closed structures at hand, we are ready to define provability in Ω-logic. That our discussion is not vacuous is the content of the following fact; in practice more delicate results are required. See [34, §4] for techniques that can easily be adapted to prove strengthened versions of Fact 7.12.

Fact 7.12. *Assume our Base Theory. Let A be a Universally Baire set. Then there are A-closed countable transitive models of* ZFC. $\square$

See [5, §2.4] for basic results about the following notion.

Definition 7.13 ($\vdash_\Omega$). Let $T \supseteq \mathsf{ZFC}$ be a theory, and let ϕ be a sentence. Then

$$T \vdash_\Omega \phi$$

iff there exists a Universally Baire set A such that

1. $L(A, \mathbb{R}) \models \mathsf{AD}^+$.
2. $A^\sharp$ exists and is Universally Baire.
3. Whenever M is a countable, transitive, A-closed model of ZFC and $\alpha \in \mathrm{ORD}^M$ is such that $M_\alpha \models T$, then $M_\alpha \models \phi$.

See [44, Chapter 10] or [47] for an introduction to AD^+.

In [44], the notion now called $\models_\Omega$ was denoted $\vdash_{\Omega^*}$ and called Ω^*-logic. Ω-logic was defined by a slight variation of Definition 7.13, namely instead of requiring that if $M_\alpha \models T$ then $M_\alpha \models \phi$ for initial segments M_α of M, this was required of M itself. The change allows for a cleaner version of the Ω-conjecture, see Conjecture 7.17. Originally, the Ω-conjecture needed to be stated in terms of Π_2 statements. The other difference between the definition given here and the one in [44] is due to the fact that Definition 7.13 is stated in ZFC and not in our Base Theory. Under our Base Theory, assumptions 1 and 2 hold automatically. These assumptions are what is required to prove the existence of appropriate A-closed structures, see [34].

One of the nicest features of $\vdash_\Omega$ is that it does not depend on the particular universe where it is considered, at least if we restrict our attention to possible generic extensions. This is the content of [44, Theorem 10.146], see [5, Theorem 2.35] for a proof.

Theorem 7.14 (Generic Invariance). *Assume our Base Theory. Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then $T \vdash_\Omega \phi$ iff for all $\mathbb{P}$, $V^{\mathbb{P}} \models T \vdash_\Omega \phi$.* $\square$

See [5, Theorem 3.3] for a proof of the following under the additional assumption of the existence of a proper class of strongly inaccessible cardinals.

Theorem 7.15 (Generic Soundness). *Let* $T \supseteq \mathsf{ZFC}$ *and let* ϕ *be a sentence. Suppose* $T \vdash_\Omega \phi$. *Then* $T \models_\Omega \phi$. □

Remark 7.16. The previous definition of $\vdash_\Omega$ required the background assumption of our Base Theory in order for Theorem 7.15 to hold. Notice that with the new definition it is stated as a ZFC result.

The Ω-conjecture is the statement that $\vdash_\Omega$ is the notion of provability associated to $\models_\Omega$ in the sense that the completeness theorem for Ω-logic holds. See [5, §3] for an interesting discussion of this conjecture.

Conjecture 7.17 (Ω-Conjecture). *Assume our Base Theory and let* ϕ *be a sentence. Then* $\mathsf{ZFC} \models_\Omega \phi$ *iff* $\mathsf{ZFC} \vdash_\Omega \phi$.

Woodin has shown that the Ω-conjecture is true unless (in a precise sense) there are large cardinal hypothesis implying a strong failure of iterability, see [44] and [45].

Definition 7.18 (Ω-consistency). Assume our Base Theory. Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then ϕ is Ω-*consistent relative to* T (and if $T = \mathsf{ZFC}$, we just say ϕ is Ω-consistent) iff for any Universally Baire set A there is an A-closed countable transitive $M \models T + \phi$.

Hence, at least as far as we can see nowadays, in order to prove that a proper class model of a Σ_2-sentence ϕ can be achieved (from large cardinals) by forcing, it suffices to show that for any Universally Baire set A, ϕ holds in an appropriate A-closed model M of ZFC. The intention of this comment is that it is not the same to prove that a sentence ϕ is forceable from an inner model than from the ground model itself. After all, ϕ may hold in forcing extensions of an inner model because that model is not sufficiently correct. For a trivial example, L admits a projective well-ordering of the reals, but such well-orderings are impossible in the presence of mild large cardinals. However, if the Ω-conjecture holds, and ϕ is Ω-consistent, then in fact ϕ can be forced over V.

Notice that any statement of the form $\exists\alpha\,(V_\alpha \models \phi)$, where ϕ is a sentence, is Σ_2, and any statement of the form $\forall\alpha\,(V_\alpha \models \phi)$, for ϕ a sentence, is Π_2. The following follows immediately:

Fact 7.19. *The statement* "$\mathsf{RVM}(\mathfrak{c})+$ There is a Δ^2_1-well-ordering of $\mathbb{R}$" *can be rendered in a* Δ_2*-way.* □

The reader should appreciate by now how powerful the Ω-conjecture is, since the witnesses to Ω-consistency of a sentence ϕ can be "fine structural-like" models, their fine structural features may be used in essential ways to establish the validity of ϕ, and nonetheless we can conclude that ϕ can be forced over the universe, without the need of any fine structural of anti-large cardinal requirements.

Since we do not know how to force a Δ^2_1-well-ordering of the reals together with $\mathsf{RVM}(\mathfrak{c})$, unless we have some nice control over the ground model itself, it was natural to attempt a proof of the Ω-consistency of this assumption. Woodin has succeeded in this attempt, and we close this section with his result and a few comments.

Theorem 7.20 (Woodin). *Assume our Base Theory. Then it is Ω-consistent that $\mathfrak{c}$ is real-valued measurable and there is a Σ^2_1-well-ordering of $\mathbb{R}$.* □

This result is proved in [47]. The idea is to use the large cardinal assumption to produce, given a Universally Baire set A, A-closed and sufficiently "fine structure-like" inner models of strong versions[18] of AD^+ over which forcing with $\mathbb{Q}_{\max}$ produces ZFC-models with a distinguished measurable cardinal. The measurable is used to produce a further extension, by forcing as in section 6. This provides us, combined with the fine structural features of the ground model, with an appropriate covering argument that can be used in place of Claim 6.3 to correctly identify enough of HOD of the ground model to obtain the desired Σ^2_1-definition. The ground model can in fact be chosen so the forcing extension itself is A-closed, and this gives the result. The covering argument rests on factoring properties of the generic embeddings derived from forcing with the nonstationary ideal, using the features that $\mathbb{Q}_{\max}$ provides.

It follows immediately that granting large cardinals, if the Ω-conjecture holds then the conclusion of Theorem 7.20 can actually be forced. The following, however, remains open (from any large cardinal assumptions).

Question 7.21. *Assume κ is measurable and* GCH *holds. Is there a forcing extension where $\kappa = \mathfrak{c}$ is real-valued measurable, and there is a Δ^2_1-well-ordering of $\mathbb{R}$?*

8. Real-valued huge cardinals

The result of this section serves a two-fold goal. It shows that $\mathsf{RVM}(\mathfrak{c})$ and a Σ^2_1, or even Σ^2_n-well-ordering of $\mathbb{R}$ for some $n < \omega$, cannot be obtained for free. It also shows that there are limits to how far the techniques of this paper can generalize.

Definition 8.1. A cardinal κ is *real-valued huge* iff there is $\lambda \geq \omega_1$ such that in $V^{\mathtt{Random}_\lambda}$ there exists an elementary embedding $j : V \xrightarrow{\prec} N$ with $\mathrm{cp}(j) = \kappa$ and such that ${}^{j(\kappa)}N \subseteq N$.

The following is clear:

Lemma 8.2. *If κ is huge, then $V^{\mathtt{Random}_\kappa} \models \kappa = \mathfrak{c}$ is real-valued huge.*

[18] These models have the form $N = L_\Gamma(\mathbb{R}, \mu)$, where μ is the restriction to N of some normal measure ν on some cardinal κ, $\mu = \nu \cap N \in N$, and Γ is a particular closure operator which also plays the role of the tree for Σ^2_1 inside the model.

Proof. Let $j : V \to M$ in V witness hugeness of κ, so ${}^{j(\kappa)}M \subseteq M$ and $\mathrm{cp}(j) = \kappa$. Set $\mathbb{Q} = \mathtt{Random}_\kappa$. Let G be $\mathbb{Q}$-generic over V, and let H be $j(\mathbb{Q})/\mathbb{Q}$-generic over $V[G]$. By Theorem 1.34, we just need to verify that in $V[G][H]$, j lifts to

$$j^* : V[G] \to M[G][H]$$

and that $V[G][H] \models {}^{j(\kappa)}M[G][H] \subseteq M[G][H]$. As usual, the lifting j^* is given by $j^*(\tau_G) = j(\tau)_{G^\frown H}$. This is well defined and elementary.

Given a sequence of names $\vec{\tau} = \langle \tau_\alpha : \alpha < j(\kappa) \rangle$ with each τ a $j(\mathbb{Q})$-name in M, the whole sequence $\vec{\tau}$ belongs to M and, therefore, $\langle (\tau_\alpha)_{G^\frown H} : \alpha < j(\kappa) \rangle \in M[G][H]$. From this the result follows. □

Having shown the consistency of real-valued hugeness of the continuum, we now point out the following observation due to Woodin:

Fact 8.3 (Woodin). *Suppose $\mathfrak{c}$ is real-valued huge. Then there are no third-order definable well-orderings of the reals.*

Proof. The same argument as for $L(\mathbb{R})$ in Theorem 2.5 works:

Towards a contradiction, let $\varphi(x, y, z)$ be a third-order formula in the language of arithmetic, and let $t \in \mathbb{R}$ be such that for some well-ordering $<$ of $\mathbb{R}$, $\varphi(r, s, t)$ holds of reals r, s iff $r < s$.

Let λ and G a $\mathtt{Random}_\lambda$-generic over V be such that in $V[G]$ there is an embedding $j : V \to N$ with $\mathrm{cp}(j) = \mathfrak{c}^V$ and ${}^{j(\mathfrak{c}^V)}N \subseteq N$. Then $\mathcal{P}(\mathbb{R})^{V[G]} \subseteq N$, since $|\mathbb{R}| = j(\mathfrak{c}^V)$ holds in N (and $\mathbb{R}^N = \mathbb{R}^{V[G]}$, since ${}^\omega N \subseteq N$.) But this means that third-order statements in the language of arithmetic, with parameters from N, are absolute between N and $V[G]$.

We are done, because by elementarity $\varphi(\cdot, \cdot, t)$ would be a third-order definition of a well-ordering of the reals in $V[G]$, but this is impossible by Corollary 2.2. □

Remark 8.4. Notice that what the proof actually shows is that if $\mathfrak{c}$ is real-valued huge and λ is as in Definition 8.1, then $V \equiv_{\underset{\sim}{\Sigma}^2_\omega} V^{\mathtt{Random}_\lambda}$, where boldface indicates that real parameters from V are allowed.

The argument of Theorem 3.1 breaks down very early when trying to adapt it to the case where κ is huge. For example, the existence of the N-generic object we called G^* cannot be ensured due to the strong closure of N.

Remark 8.4 suggests the natural question of whether generic invariance of Σ^2_ω with respect to "$\mathfrak{c}$ is real-valued huge" holds. This seems somewhat delicate, since there does not seem to be a natural counterpart to Solovay's Fact 1.27 for preservation of real-valued hugeness. The hypothesis is by no means intended to be optimal. For example, it is not clear whether the natural real-valued version of $\mathcal{P}^2(\kappa)$-measurability of κ for $\kappa = \mathfrak{c}$ suffices to rule out the existence of third-order definable well-orderings of $\mathbb{R}$.

As expected, real-valued hugeness is a serious large cardinal assumption, strictly stronger than real-valued measurability. Here we content ourselves with some easy observations and a remark:

Fact 8.5. *If κ is real-valued huge, then there are weakly inaccessible cardinals larger than κ.*

Proof. Let λ be as in Definition 8.1, and in $V^{\texttt{Random}_\lambda}$, let $j : V \to N$ be the witnessing embedding. Then $N \models j(\kappa)$ is real-valued measurable, so in particular $N \models j(\kappa)$ is weakly inaccessible.

But $V[G] \models {}^{j(\kappa)}N \subseteq N$, so $j(\kappa)$ *is* weakly inaccessible in $V[G]$, and therefore in V.

As usual, the proof actually shows that there are fixed points of the weakly Mahlo hierarchy (see [26, after Proposition 1.1]), etc., above κ. □

Theorem 8.6. *If $\kappa \leq \mathfrak{c}$ is real-valued huge, then the real-valued measurable cardinals are unbounded below κ. In fact, for a witnessing probability ν,*

$$\nu(\{\, \alpha < \kappa : \mathsf{RVM}(\alpha) \,\}) = 1.$$

Proof. As before, let λ be as in Definition 8.1. Let $\varphi : \texttt{Random}_\lambda \to [0,1]$ be the 'probability measure' associated to $\texttt{Random}_\lambda$, fix a $\texttt{Random}_\lambda$-generic G over V and, in $V[G]$, let $j : V \to N$ witness real-valued hugeness of κ.

By Fact 1.27, $\mathsf{RVM}(\kappa)$ holds in $V[G]$. Let $\hat{\nu} : \mathcal{P}(\kappa) \to [0,1]$ be a witness. Notice that $[0,1] \in N$. Since in $V[G]$, ${}^{j(\kappa)}N \subseteq N$ and $|\mathcal{P}(\kappa)| = 2^\kappa \leq j(\kappa)$, then in particular $\hat{\nu} \in N$. Thus, $N \models \mathsf{RVM}(\kappa)$.

Since G was arbitrary, $\varphi[\![\kappa \in \dot{j}(\{\, \alpha : \mathsf{RVM}(\alpha) \,\})]\!] = \mathtt{1}$, where $\dot{j}$ denotes a term for an embedding witnessing real-valued hugeness of κ.

In V, let $\nu : \mathcal{P}(\kappa) \to [0,1]$ be defined as usual by $\nu(A) = \varphi[\![\kappa \in \dot{j}(A)]\!]$. Then ν is as required.

As usual, this proof actually gives that κ is limit of real-valued measurable cardinals that also concentrate on real-valued measurable cardinals that concentrate on real-valued measurable cardinals, etc. □

Real-valued huge cardinals imply the existence of inner models for Woodin cardinals. In the presence of measurable cardinals this is an immediate consequence of the following result of Steel. It appears as [40, Theorem 7.1] under the stronger assumption that Ω is measurable.

Theorem 8.7 (Steel). *Suppose $V_\Omega^\sharp$ exists, and let G be $\mathbb{P}$-generic over V for some $\mathbb{P} \in V_\Omega$. Suppose that in $V[G]$ there is a transitive class M and an elementary embedding*

$$j : V \to M \subseteq V[G]$$

with $\mathrm{cp}(j) = \kappa$ and such that $V[G] \models {}^{<j(\kappa)}M \subseteq M$. Then the K^c-construction reaches a non-1-small level[19]. □

[19] I.e., $M_1^\sharp$, the sharp for a proper class fine structural inner model with a Woodin cardinal, exists.

In fact, much more follows from this hypothesis. For example, it is straightforward to improve the argument leading to Theorem 8.6 to a proof of the fact that there is a 'probability measure' $\nu : \mathcal{P}(\mathfrak{c}) \to [0,1]$ such that $\nu(\{\,\alpha : \alpha$ is real-valued almost huge $\}) = 1$. Here, a cardinal κ is called real-valued almost huge iff there is a $\lambda \geq \omega_1$ such that in $V^{\mathtt{Random}_\lambda}$ there is an embedding $j : V \to N$ with $\mathrm{cp}(j) = \kappa$ and such that $V^{\mathtt{Random}_\lambda} \models {}^{<j(\kappa)}N \subseteq N$.

Using his technique of the core model induction, Woodin has shown:

Theorem 8.8 (Woodin). *If there is a real-valued almost huge cardinal, then* $\mathsf{AD}^{L(\mathbb{R}\cup\{\mathbb{R}^\sharp\})}$ *holds.* □

For more on real-valued huge cardinals and a strengthening of the above result, see [12].

Remark 8.9. Anti-definability results can also be achieved by fine structural arguments starting with $V = L[\mu]$.

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