Arch. Math. Logic (2006) 45:783–793
DOI 10.1007/s00153-006-0002-6

Mathematical Logic

Projective well-orderings of the reals

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Received: 3 August 2004 / Revised: 9 September 2004 /
Published online: 17 June 2006

Abstract If there is no inner model with ω many strong cardinals, then there is a set forcing extension of the universe with a projective well-ordering of the reals.

Keywords Projective well-orderings · Core models · Strong cardinals

Mathematics Subject Classification (2000) 03E15 · 03E35 · 03E45 · 03E55

1 Introduction

The goal of this paper is to prove a result, Theorem 2, illustrating the strength (rather, lack thereof) of the hypothesis that the reals admit a projective well-ordering. The reader should interpret this as saying that the lack of significant large cardinal structure in the universe allows for pathological well-orderings. This result appears in Chapter 2 of the first author's dissertation [1] written under the supervision of John Steel and Hugh Woodin at U.C. Berkeley.

Theorem 2 is a corollary of computations due to the second author, and it was obtained by him independently. The version we present provides optimal

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complexity bounds, and in this form is due to the second author. In what follows, by forcing we mean set-forcing. By a real we mean a subset of ω, although any of the usual renderings of $\mathbb{R}$ would work just as well.

2 Below sharps

The following is most likely folklore but we were unable to locate a reference:

Theorem 1. *Suppose V is not closed under sharps. Then there is a forcing extension of the universe with a $\utilde{\mathbf{\Sigma}}^1_2$-well-ordering of the reals.*

Corollary 1. *The conclusion of Theorem 1 holds in either of the following situations:*

1. $\omega_1 = \omega_1^{L[x]}$ *for some real x.*
2. *K exists and is not closed under sharps.*

Proof. If V is closed under sharps, then so is K, and clearly $\omega_1^{L[x]} < \omega_1$ for any real x. □

The argument for Theorem 1 is actually quite easy and resembles that in [11], but without some of its complications. We decided to include it, since the proof of Theorem 2 builds on this one.

Proof of Theorem 1. Let X be (a set of ordinals) such that $X^\sharp$ does not exist. By Jensen's covering lemma there is $\delta > \sup X$ such that $\delta^+ = (\delta^+)^{L[X]}$. Force to make δ countable while preserving δ^+, and let r be a real coding X and δ. Then $\omega_1 = \omega_1^{L[r]}$. Fix λ singular strong limit. Then λ is singular in $L[r]$, $\nu := \lambda^+ = (\lambda^+)^{L[r]}$, and $2^\lambda = \lambda^+$, as otherwise $r^\sharp$ exists. It follows that there is $A_1 \subseteq \lambda^+$ such that $L_\nu[r, A_1] = H_\nu$. Let G be $\mathtt{Coll}(\omega, \lambda)$-generic over V, and work in $V[G]$. Let $A_2 \subseteq \omega$ code G. Notice $\nu = \omega_1^{V[G]}$.

It follows that $\mathrm{HC} = L_\nu[r, A_1, A_2]$, as an easy consequence of the λ^+-cc of the forcing: any name for an element of $\mathrm{HC}^{V^{\mathtt{Coll}(\omega,\lambda)}}$ appears as an element of $(H_\nu)^V$.

Let $V_1 = V[G]$. In V_1 let $A \subseteq \lambda^+$ code r, A_1, A_2. We apply the almost disjoint forcing technique of Jensen and Solovay to an extension of V_1.

Claim. There is a set forcing extension of V_1 preserving ν where there is a real t such that $\mathrm{HC} = L_\nu[t]$.

Proof. Work inside V_1. In $L[r]$ there is a definable sequence of λ^+ subsets of λ. Any reasonable such sequence in fact is definable over $L_\nu[r]$ and, moreover, for a club set of $\gamma < \nu$ the same definition over $L_\gamma[r]$ gives the first γ-many terms of the sequence.

Let $\hat{s}$ be a real coding λ. Then in $L[r, \hat{s}]$ we can easily define from the $L[r]$-sequence an ω_1-sequence of almost disjoint reals.

Let $\mathcal{A} = \langle q_\alpha : \alpha < \omega_1 \rangle$ be such an $L_{\omega_1}[r, \hat{s}]$-definable sequence. Recall that $\mathrm{HC} = L_{\omega_1}[r, A]$.

Let $\mathbb{Q}$ be the usual forcing for coding A by a real, using $\mathcal{A}$, namely

$$\mathbb{Q} = \left\{(\mathsf{s}, \mathsf{F}) : \mathsf{s} \in 2^{<\omega} \text{ and } \mathsf{F} \in \mathcal{P}_\omega(\omega_1)\right\},$$

ordered by

$$(\mathsf{s}_1, \mathsf{F}_1) \leq (\mathsf{s}_2, \mathsf{F}_2) \iff \begin{cases} \mathsf{s}_1 \supseteq \mathsf{s}_2, \\ \mathsf{F}_1 \supseteq \mathsf{F}_2 \text{ and} \\ \forall \alpha \in \mathsf{F}_2 \cap A \left(\left(\mathsf{s}_1^{-1}\{1\}\right) \setminus \mathrm{dom}(\mathsf{s}_2)\right) \cap q_\alpha = \emptyset). \end{cases}$$

It is easy to see, and well known (see for example [6, §II.2]), that $\mathbb{Q}$ is σ-centered and that if G_1 is $\mathbb{Q}$-generic over V_1, then

$$r_{G_1} := \bigcup \left\{\mathsf{s} : \exists \mathsf{F}\big((\mathsf{s}, \mathsf{F}) \in G_1\big)\right\} \in (2^\omega)^{V_1[G_1]},$$

G_1 is definable from r_{G_1} in $V_1[G_1]$, and for all $\alpha \in \omega_1$, $\alpha \in A$ iff $|r_{G_1} \cap q_\alpha| < \omega$.

Fix such a G_1. We claim that $V_1[G_1]$ is as wanted, and that $t = \{2n : n \in \hat{s}\} \cup \{2n + 1 : n \in r_{G_1}\}$ serves as a witness. Namely, $L_{\omega_1}[r, A] \subset L[t]$, since $\hat{s} \in L[t]$, $\mathcal{A}$ is definable in $L[r, \hat{s}]$, and A is definable from $\mathcal{A}$ and r_{G_1}. Therefore $\mathrm{HC} \subseteq L[t]$, since $\mathbb{Q}$ is ccc. Clearly, $L_{\omega_1}[t] \subseteq \mathrm{HC}$, and we are done. $\triangle$

Let V_2 be the universe obtained in the claim. Then in V_2 the reals admit a $\Delta^1_2(t)$-well-ordering, namely, the natural wellordering of $\mathbb{R}^{L[t]}$. $\square$

Remark 1. In fact, if V is not closed under sharps, then the method of Schindler [11] gives a stationary set preserving (in fact, reasonable) forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$ there is a $\utilde{\mathbf{\Delta}}^1_2$-well-ordering of $\mathbb{R}$.

3 Below ω strong cardinals

Theorem 2. *Suppose there is no inner model with ω strong cardinals. Then there is a set forcing extension of V with a projective well-ordering of the reals. In fact, the core model K exists, and if n is the number of strong cardinals in K, then there is an extension of V with a $\utilde{\mathbf{\Delta}}^1_{n+3}$-well-ordering of $\mathbb{R}$. The well-ordering can actually be obtained to be $\utilde{\mathbf{\Delta}}^1_2$ if $n = 0$ and V is not closed under sharps.*

Proof. The idea of the proof is to obtain a model where CH holds and there is a projectively definable ω_1-sequence of almost disjoint reals, from which by judicious use of almost disjoint forcing, as above, we can define a well-ordering of $\mathbb{R}$.

More carefully, we look for a model M of enough set theory containing all the reals and such that

$$M \models \mathsf{CH} + \text{There is a projective well-ordering of } \mathbb{R}.$$

We arrange by our use of almost disjoint forcing that $M \cap \mathrm{HC}$ is itself projectively definable, thus obtaining the desired well-ordering.

Let K be the core model, and let $\mathcal{E}$ be the coherent sequence of extenders of K, $\mathcal{E} = E^K$, so $K = L[\mathcal{E}]$. The model we will work with is $K[r] = L[\mathcal{E}, r]$, where r is a real. We will arrange things so $K[r] \supseteq \mathrm{HC}$ (and CH holds). That $K[r]$ or, rather, a sufficiently long initial segment of $K[r]$ is projective in the codes if there are no inner models of ω strong cardinals follows from results of Schindler [12], Hauser and Schindler [5] and Hauser and Hjorth [4].

Now we proceed to the details. We use [5,12], so mice and premice are in the Jensen rather than the Mitchell–Steel sense.

Suppose that there is no inner model with ω many strong cardinals. Then $0^\P$ does not exist[1], and this implies by [12] that K does. K is definable, and generically invariant in the sense that for any poset $\mathbb{P}$, $K^V = K^{V^{\mathbb{P}}}$.

Claim. Let W be a universal weasel (so $\mathrm{ORD}^W = \mathrm{ORD}$) and suppose $W \models \beta$ is a strong cardinal. Then

$$W \models \beta \text{ is a strong cardinal, as witnessed by extenders on } \mathcal{E}.$$

Proof. This is most likely folklore and archeology reveals variations of it in print. See, for example, [14, Theorem 8.14]. For the case that concerns us, the result follows from the argument given in [4, Lemma 1.5], together with the realization that below $0^\P$, the references to the measurable Ω can be dispensed with in that proof.

Here is a brief sketch:

Since $0^\P$ does not exist, $W \models$ I am iterable and there is an embedding $j : K \to W$. Since $K \models V = K$, we may assume $W = K$, and work inside W.

Suppose β is a strong cardinal, and let $\alpha > \beta^+$ be a cardinal. Let K_1 witness the very soundness of a sufficiently long initial segment of K, say $K\|\alpha$. Let E be an extender witnessing β is strong past α, and consider the ultrapower embedding $\pi_E : K \to \mathrm{Ult}(K, E)$. In virtue of the inductive definition of K, we have that $K^{\mathrm{Ult}(K,E)}\|\alpha = K\|\alpha$. Let $K_2 = \pi_E(K_1)$. Then $K_1\|\alpha = K_2\|\alpha$, and K_2 is a soundness witness for $K\|\beta$ but not for $K\|\beta + 1$.

Compare K_1 and K_2, so they iterate to a common model K^*. Let $\pi_1 : K_1 \to K^*$ and $\pi_2 : K_2 \to K^*$ be the iterations arising from the comparison. Then a standard argument using the definability property shows that $\mathrm{cp}(\pi_1) = \beta$. It follows that in the K_1-to-K^* side of the comparison, an extender with critical point β was used, and the agreement of K_1 and K_2 implies that its length is greater than α.

[1] $0^\P$, zero-hand-grenade, was introduced in the second author's Habilitationsschrift, published as Schindler [12], where it is shown that K exists if $0^\P$ does not exist. The existence of $0^\P$ is equivalent to the existence of indiscernibles for a proper class model with a proper class of strong cardinals.

It follows from [12, Corollary 1.3] together with coherency that

$$K_1 \models \beta \text{ is strong up to } \alpha, \text{as witnessed by extenders} \\ \text{on the } E^{K_1} - \text{sequence,}$$

and therefore the same holds in K. $\triangle$

From now on, by strong cardinal we understand strong cardinal, as witnessed by extenders on the sequence.

Suppose that there are exactly n strong cardinals in K, and (by Theorem 1) that V is closed under sharps. We claim that there is a set forcing extension of the universe admitting a $\underset{\sim}{\mathbf{\Delta}}^1_{n+3}$-well-ordering of the reals.

Set $\delta = 0$ if $n = 0$, and otherwise let δ be the largest K-cardinal κ such that $K \models \kappa$ is a strong cardinal.

Claim. There is a strong limit singular cardinal λ such that $\lambda^+ = (\lambda^+)^K$, $\delta < \lambda$, and for all $\kappa < \lambda$, κ is a strong cardinal in $K\|\lambda$ iff κ is a strong cardinal in K.

Proof. By the covering lemma ([12, Theorem 8.18], which really follows from [8,9]), for any $\beta \geq \omega_2$, $\mathrm{cf}(\beta^{+K}) \geq |\beta|$. In particular, for any singular λ, $\lambda^+ = (\lambda^+)^K$.

Now the result is easy. Let $o(\alpha)$, for α an ordinal, denote the Mitchell order of α in K,

$$o(\alpha) = \mathrm{ot}\left\{\nu : \mathcal{E}_\nu \text{ is total on } K \text{ and } \mathrm{cp}(\mathcal{E}_\nu) = \alpha\right\},$$

so $o(\alpha) \in \mathrm{ORD}$ iff α is not a strong cardinal. Let $\lambda > \delta$ be strong limit singular and closed under $\beta \mapsto o(\beta)$. We are done, once we verify that for β a strong cardinal in K, the lengths of the extenders in $K\|\lambda$ with critical point β are unbounded in λ. But this is clear: If β is a strong cardinal in K, then it is a strong cardinal as witnessed by extenders on $\mathcal{E}$, and [12, Corollary 1.3] shows that for any cardinal $\alpha > (\beta^+)^K$, both the lengths and the indices of extenders on $\mathcal{E}$ witnessing that β is $< \alpha$-strong are cofinal in α. In particular, this holds for λ, so $K\|\lambda \models \beta$ is a strong cardinal. $\triangle$

Fix λ as in the claim. The key to most results involving simply definable well-orderings of the reals is to use almost disjoint forcing. This is what we do here, in a fine structural context. First, we need a projectively definable uncountable sequence of almost disjoint reals.

Claim. There is a set forcing extension of the universe that collapses λ to ω while preserving $\nu := \lambda^+$ and where, moreover, $\mathrm{HC} = L_\nu[\mathcal{E}, A]$ for some $A \subseteq \lambda^+$.

Proof. By forcing with $\mathtt{Coll}(\nu, 2^\lambda)$ if necessary, we may assume $2^\lambda = \lambda^+$. Now proceed as in the proof of Theorem 1. $\triangle$

Call V_1 the universe obtained in the claim.

Claim. There is a set forcing extension of V_1 preserving $\nu(=\omega_1^{V_1})$ where there is a real r such that $\mathrm{HC} = L_\nu[\mathcal{E}, r]$.

Proof. Exactly as for the claim in the proof of Theorem 1, with K instead of $L[r]$. $\triangle$

Recall that $\delta < \lambda$ is the largest strong cardinal of K. Let V_2 be the universe obtained in the claim, and work in V_2. The following key lemma provides the optimal complexity for the well-ordering we are to obtain.

Lemma 1. *$L_\nu[\mathcal{E}]$ is $\Delta^1_{n+3}(s)$ in the codes, where $s \in \mathbb{R}$ codes $K\|\delta$.*

Proof. This follows from Schindler's arguments in [5]. See Theorem 3.6 there and the comment after the proof of Theorem 3.5. $\triangle$

Remark 2. In detail, the claim states that $\{r \in \mathbb{R} : r \text{ codes } \mathcal{M} \lhd \mathcal{J}^K_{\omega_1^V}\}$ is projective. As usual, $\bar{P} \trianglelefteq \bar{Q}$ iff $\bar{P}$ is an initial segment of $\bar{Q}$.

It now follows that in V_2 the reals admit a $\Delta^1_{n+3}(r, s)$-well-ordering, where r and s are as above. Namely, $K[r] = L(K \cup \{r\})$ so $\mathbb{R}^{V_2} = \mathbb{R}^{K[r]}$ admits a natural well-ordering, derived from the order of constructibility (closing under terms for Gödel operations) and the natural well-ordering of K. More carefully, $K[r] = K[\hat{s}][A][r]$, but $\hat{s}$ is recursive in r and A is easily definable from r and $\mathcal{A}$, which in turn is easily definable in $K[\hat{s}]$ from $\hat{s}$ and a sequence of sets locally definable in $K\|\omega_1$. Unfolding this construction, the terms produced by Gödel operations only require to be (hereditarily) evaluated in elements of $K\|\omega_1$ and the real r. So we obtain a well-ordering by only listing those terms that produce reals, and avoiding repetitions. Since the terms are naturally well-ordered, we only need to see how difficult it is to identify $K\|\omega_1$ inside $K[r]$. Lemma 1 tells us that it is $\Delta^1_{n+3}(s)$, and we are done. $\square$

Remark 3. If there is no inner model with a strong cardinal, then by [11] there is a reasonable forcing extension of the universe with a $\utilde{\boldsymbol{\Delta}}^1_3$- well-ordering of $\mathbb{R}$. It is not known whether Theorem 2 can be obtained in general (when there is at least one but only finitely many strong cardinals in K) if we restrict ourselves to extensions by set sized reasonable forcing.

The theorem can be strengtened in a straightforward way: say that a set x is *adequate* iff there is no inner model with x as an element and ω many strong cardinals above x (i.e., above $\mathrm{rk}(x)$). If x is adequate, then K_x, the core model built over x using x-mice, exists. Thus, we can weaken the hypothesis of Theorem 2 to: suppose there is an adequate x. And choose n as in the conclusion of the theorem to be minimal such that there is an adequate x with $K_x \models$ There are exactly n strong cardinals (above x). This may indeed reduce the complexity of the forced well-ordering. For example, consider $V = L[r]$, where we start with K containing, say, exactly five strong cardinals, and r is a real obtained by doing Jensen coding over K, so $K[r] = L[r]$. Then $K^{L[r]}$ exists and has exactly five strong cardinals (by a straightforward comparison argument—it must in fact be the case that $K = K^{L[r]}$) but $K_r = L[r]$ is not even closed under sharps.

4 The strength of projective well-orderings

In this section we show that the complexity of the well-orderings obtained in Theorem 2 is in general best possible. This follows from arguments due to Woodin, see [13] for further details.

Obviously, no Σ^1_2-well-ordering is possible, unless $\mathbb{R} \subset L$. By [7], no $\underset{\sim}{\mathbf{\Sigma}}^1_2$-well-ordering is possible, unless $\mathbb{R} \subset L[r]$ for some real r. In particular, CH must hold.

Fact 1. *If V is closed under sharps, then there is no set generic extension with a $\underset{\sim}{\mathbf{\Sigma}}^1_2$-well-ordering of $\mathbb{R}$.*

Proof. If V is closed under sharps, then so is any set generic extension. Let G be generic over V for some forcing notion $\mathbb{P}$. For any real r in $V[G]$, $r^\sharp \notin L[r]$, so by Mansfield's result [7] there is no $\Sigma^1_2(r)$-well-ordering of $\mathbb{R}^{V[G]}$ in $V[G]$. □

Woodin's result correlates levels of setgeneric projective absoluteness with the existence of inner models for strong cardinals. The base case is slightly different from the rest, and we present in some detail a proof of it that does not make use of the Martin–Solovay tree.

Let Γ be a class of formulas. By Γ-generic absoluteness we mean the assertion that whenever $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration of set forcings, $x \in \mathbb{R}^{V^{\mathbb{P}}}$, and $\phi \in \Gamma$, then $V^{\mathbb{P}} \models \phi(x)$ iff $V^{\mathbb{P}*\dot{\mathbb{Q}}} \models \phi(\check{x})$.

Theorem 3 (Martin, Solovay; Woodin). *The following are equivalent:*

1. *V is Σ^1_3-absolute.*
2. *V is closed under sharps.*

Proof. (*1.* $\Rightarrow$ *2.*) Suppose X is a set of ordinals such that $X^\sharp$ does not exist, so there is $\delta > \sup X$ such that $\delta^+ = (\delta^+)^{L[X]}$. In $V^{\mathtt{Coll}(\omega,\delta)}$, let x be a real coding X and δ, so $\omega_1 = \omega_1^{L[x]}$. Notice that this is a $\Pi^1_3(x)$-statement, and that it obviously fails in $V^{\mathtt{Coll}(\omega,\delta^+)}$. It follows that V is not Σ^1_3-absolute.

(*2.* $\Rightarrow$ *1.*) The usual way of proving this is via the Martin–Solovay tree. We provide here a slightly different argument.

Suppose that V is closed under sharps. Recall that for a set of ordinals X, the existence of $X^\sharp$ is equivalent to the assertion that there is an X-mouse of the form $\mathcal{M} = (J_\alpha[X], \in, X, U)$ such that $\mathcal{M} \models$ There is a largest cardinal κ and U is a κ-complete normal measure on κ. We will refer to such a (unique) mouse as $X^\sharp$ in what follows.

Let φ be Π^1_2. We claim that if there is a forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$ there is a real x such that $V^{\mathbb{P}} \models \varphi(x)$, then there is such a real in V. By Shoenfield's absoluteness, it follows that V is Σ^1_3-absolute.

Fix such a $\mathbb{P}$ which we may assume is a set of ordinals, so we can choose X such that $\mathbb{P} \in X^\sharp$. Let $\mathcal{T}$ be the tree of attempts to build a quintuple $(N, \mathbb{Q}, g, y, \pi)$ such that

(a) N is a countable transitive premouse.
(b) $\mathbb{Q} \in N$ is a partial order.
(c) g is $\mathbb{Q}$-generic over N.
(d) y is a real in $N[g]$ and $N[g] \models \varphi(y)$.
(e) $\pi : N \to X^\sharp$ is elementary, and $\pi(\mathbb{Q}) = \mathbb{P}$.

Then $\mathcal{T}$ is *ill-founded* in $V^{\mathbb{P}}$, and therefore in V. Let $(N, \mathbb{Q}, g, y, \pi) \in [T]^V$. Then (because of π) N is iterable. $N[g]$ is a Y-premouse (for some Y), and the iteration of N lifts to the iteration of $N[g]$, so $N[g]$ is actually iterable. Since $N[g] \models \varphi(y)$, there is therefore a class-sized model $\mathcal{N}$ such that $\mathcal{N} \models \varphi(y)$. By Shoenfield's absoluteness, $V \models \varphi(y)$ as well, and we are done. □

The corresponding result on absoluteness of higher levels of the projective hierarchy is given by:

Theorem 4 (Woodin). *If there are $n > 0$ strong cardinals and κ is larger than all of them, then Σ^1_{n+3}-generic absoluteness holds in $V^{\mathtt{Coll}(\omega, 2^{2^\kappa})}$.* □

This is proved by induction, starting with the base case provided by Theorem 3. The key is the following theorem, see ([13], Section 3):

Theorem 5 (Woodin). *Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is λ-universally Baire as witnessed by T, T'. Let κ be a strong cardinal, with $2^{2^\kappa} < \lambda$. Then in $V^{\mathtt{Coll}(\omega, 2^{2^\kappa})}$, $p[A]$ is λ-universally Baire.* □

Corollary 2. *Assume there are $n > 0$ strong cardinals. Then there is a forcing extension of V such that there is no $\boldsymbol{\Sigma}^1_{n+2}$-well-ordering of the reals in any further set generic extension.*

Proof. Let κ be larger than the first n strong cardinals. The statement "There is a $\boldsymbol{\Sigma}^1_{n+2}$-well-ordering of $\mathbb{R}$" is Σ^1_{n+3} and it *fails* in $V^{\mathtt{Coll}(\omega, 2^{2^\kappa})}$, since $\mathtt{Coll}(\omega, 2^{2^\kappa})$ is homogeneous. □

Corollary 3. *The following theories are equiconsistent (over* ZFC*):*

1. *There are ω strong cardinals.*
2. *There is no set forcing extension of V with a projective well-ordering of the reals.*
3. *Projective absoluteness.*

Proof. (Con(ZFC + *1.*) $\Rightarrow$ Con(ZFC + *3.*)) By Woodin's Theorem 4 if $\lambda = \sup_n \kappa_n$ where $\kappa_0 < \kappa_1 < \ldots$ are strong cardinals, then projective absoluteness holds in $V_1 = V^{\mathtt{Coll}(\omega, \lambda)}$.

(*3.* $\Rightarrow$ *2.*) Since homogeneous forcing destroys any projective well-ordering, projective absoluteness ensures that no set forcing extension of V_1 can have a projective well-ordering of $\mathbb{R}$.

(Con(ZFC + *2.*) $\Rightarrow$ Con(ZFC + *1.*)) This is immediate from Theorem 2. □

The equiconsistency of *3.* and *1.* in Corollary 3 was already known, and it is the content of Hauser's Habilitationsschrift, see [3].

5 Reasonable extensions

Recall that a forcing $\mathbb{P}$ is *reasonable* iff for all $\alpha \geq \omega_1$, $\mathcal{P}_{\omega_1}(\alpha)^V$ is stationary in $\mathcal{P}_{\omega_1}(\alpha)^{V^{\mathbb{P}}}$. In [11] the second author showed that if there is no inner model with a strong cardinal, then there is a reasonable forcing extension of the universe where the reals admit a $\underset{\sim}{\mathbf{\Sigma}}^1_3$-well-ordering. This was based on a reshaping result in the lack of strong cardinals, that we now generalize slightly.

Assume that K exists, and $K \models$There is exactly one strong cardinal. Let κ denote the strong cardinal in K, and suppose $\kappa < \omega_1^V$. Then we can proceed as in [11]: Find in V a singular cardinal δ of cofinality ω_1^V whose successor is computed correctly in K, and such that the only overlaps in the sequence of extenders $\mathcal{E} = E^K$ come from extenders with critical point κ. Then we can collapse δ to ω_1^V and find in the extension a set $A \subseteq \omega_1$ such that $H_{\omega_2} = K\|\omega_2[A](= J_{\omega_2}[\mathcal{E}, A])$. We may assume $A = A_o \oplus A_e$ with A_o coding $K\|\delta$.

Then $K\|\omega_2$ is the stack of mice $\mathcal{M}$ which collapse to δ and such that only extenders that overlap δ have critical point κ, i.e., the stack of all those sound $\mathcal{M}$ such that the phalanx $(\langle K, \mathcal{M}\rangle, \kappa^+)$ is iterable, and $\rho_\omega(\mathcal{M}) \leq \delta$.

Then we can "reshape": Call an ordinal ξ *typical* iff $A_o \cap \xi$ codes a model $\bar{K}$ such that there is a sound $\mathcal{M} \trianglerighteq \bar{K}$ such that $(\langle K, \mathcal{M}\rangle, \kappa^+)$ is iterable and $\rho_\omega(\mathcal{M})$ is at most the height of $\bar{K}$.

Consider the poset $\mathbb{P}$ consisting of pairs (p, c) where for some countable α, c is a club in α consisting of typical ordinals, and $p : \alpha \to 2$ is such that for all $\xi \in c \cup \{\alpha\}$ and definably over $\mathcal{M}[A \cap \xi, p\restriction_\xi]$, the statement "$\xi$ is countable" holds. $\mathbb{P}$ is ordered as usual, and usual arguments show it is reasonable.

This forcing adds a set B. Let $C = A \oplus B$. Now use almost disjoint forcing to add a real a such that for any $\xi < \omega_1^V$, $\xi \in C$ iff a is almost disjoint from a_ξ, where $\langle a_\xi : \xi < \omega_1^V\rangle$ is the almost disjoint sequence of reals such that a_ξ is definably over $\mathcal{M}[C \cap \xi]$ the first subset of ω almost disjoint from each a_η, $\eta < \xi$. Here, $\mathcal{M}$ is as above. The sequence is well defined and independent of the choice of mice $\mathcal{M}$ since (by the standard coiteration argument) any two such $\mathcal{M}$ are lined up.

In the resulting extension, the reals admit a $\underset{\sim}{\mathbf{\Sigma}}^1_4$-well-ordering: Let "$\mathcal{M}$ is jointly iterable with K" denote the statement that $\mathcal{M}$ is sound and

$$(\langle K, \mathcal{M}\rangle, \kappa^+)$$

is iterable. This is a Π^1_3-condition on $\mathcal{M}$. Then we can define a well-ordering by stating that $x < y$ iff there is a stack of mice $\mathcal{M}$ as above such that for some $\mathcal{M}$ in the stack, $x, y \in \mathcal{M}$ and $x <_{\mathcal{M}} y$. This is a $\Sigma^1_4(a)$-well-ordering.

The same idea works if there are finitely many strong cardinals in K and all of them are below ω_1^V. However, even if K has exactly one strong cardinal, we do not know if there is a reasonable reshaping of the universe giving a similar well-ordering of the reals of the extension if the cardinal lies above ω_1^V.

In the case where K has exactly one strong cardinal κ, and $\kappa = \omega_1^V$, then we can extend the argument above, as long as $(\kappa^+)^K = \omega_2^V$.

6 Some questions

Theorem 2 produces boldface well-orderings, so there is an obvious question whose answer is still missing, for example:

Question 1. Assume $\neg 0^\sharp$. Is there a (set) forcing extension of V with a (lightface) Σ^1_3-well-ordering of $\mathbb{R}$?

As observed by the second author in [11], if $\mathbb{R}$ is closed under sharps, there is no inner model with a strong cardinal, and there is a $\Sigma^1_3(r)$-well-ordering of $\mathbb{R}$ for some real r, then $\mathbb{R} = \mathbb{R}^{K_r}$ where K_r is the core model built over r using r-mice. It follows in particular that CH must hold. The result does not hold in the presence of strong cardinals: If a is a real, K_a exists, and does not contain inner models with Woodin cardinals, then $\mathbb{R}^{K_a}$ admits a $\Sigma^1_3(a)$-well-ordering, even if the reals of no forcing extension of V do.

It is also not the case that CH must hold if there is an inner model with $0 < n < \omega$ strong cardinals, and there is no inner model with $n+1$ strong cardinals, but there is a $\utilde{\mathbf{\Sigma}}^1_{n+3}$-well-ordering of $\mathbb{R}$. To see this, let K denote the least inner model with n strong cardinals. Add ω_2 many Cohen reals. Then force a well-ordering of the reals à la [2]. Both forcings have the c.c.c. and in the extension there is a $\utilde{\mathbf{\Sigma}}^1_4$-well-ordering of $\mathbb{R}$, for the same reason as there is a $\utilde{\mathbf{\Sigma}}^1_3$-well-ordering of $\mathbb{R}$ if the same construction is carried out over L, as argued in [2].

Question 2. Suppose $n > 0$, there is an inner model with n strong cardinals, and there is no inner model with $n+1$ strong cardinals. Is there a reasonable set generic extension of the universe with a $\utilde{\mathbf{\Sigma}}^1_{n+3}$-well-ordering of $\mathbb{R}$?

A much deeper property of theory *2.* in the statement of Corollary 3 above seems untractable with current fine structural techniques.

Question 3. Is it true that projective absoluteness holds iff there is no forcing extension of V with a projective well-ordering of the reals?

This seems to require a level-by-level argument correlating projective absoluteness and the existence of Universally Baire tree representations for projective sets.

Acknowledgements Research partially supported by the FWF, project # P16334-N05

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