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PROJECTIVE WELL-ORDERINGS AND BOUNDED FORCING AXIOMS

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Abstract. In the absence of Woodin cardinals, fine structural inner models for mild large cardinal hypotheses admit forcing extensions where bounded forcing axioms hold and yet the reals are projectively well-ordered.

The main result of this paper is an improved and revised version of a theorem in Chapter 2 of the author's dissertation [5] written under the supervision of John Steel and Hugh Woodin at U.C. Berkeley.

§1. Introduction. By MA we mean in this paper $\mathsf{MA} + \neg\mathsf{CH}$. Harrington [15, Theorem B] showed that MA is consistent with the existence of a $\undertilde{\Sigma}^1_3$-well-ordering of the reals. More precisely, there is a forcing extension of L where MA holds and whose reals admit a well-ordering of the claimed complexity. Harrington's result is optimal in the sense that $\undertilde{\Sigma}^1_2$-well-orderings are incompatible with MA; this is an immediate corollary of the following classical result, see [17, Theorem 25.39]:

Theorem 1.1 (Mansfield). *Let $a \in \mathbb{R}$. The reals admit a $\Sigma^1_2(a)$-well-ordering iff $\mathbb{R} \subset L[a]$. In particular, if there is a $\undertilde{\Sigma}^1_2$-well-ordering of $\mathbb{R}$, then CH holds.* □

There is a sense in which Harrington's theorem is *not* optimal, namely the well-ordering in [15] is not lightface. This is not really an obstacle:

Theorem 1.2 (Friedman [10, Theorem 8.51]). *There is a forcing extension of L that preserves ω_1 in which MA holds and there is a Σ^1_3-well-ordering of the reals.* □

In this paper we strengthen Harrington's result in a different direction.
Recall:

Definition 1.3 ($\mathsf{SPFA}(\mathfrak{c})$). The semiproper forcing axiom holds restricted to posets of size at most $\mathfrak{c}$, i.e., if $|\mathbb{P}| \leq \mathfrak{c}$, $\mathbb{P}$ is semiproper, and $\mathscr{D} \subseteq \mathscr{P}(\mathbb{P})$ is a collection of at most $\aleph_1$ many dense subsets of $\mathbb{P}$, then there is a $\mathscr{D}$-*generic* filter $\mathscr{G} \subseteq \mathbb{P}$, i.e.,

$$\forall D \in \mathscr{D}(D \cap \mathscr{G} \neq \emptyset).$$

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Definition 1.4 (Goldstern, Shelah [13, Definition 0.1]).

1. The bounded semiproper forcing axiom, BSPFA, holds iff whenever $\mathbb{P}$ is semiproper and $\mathscr{D}$ is a collection of at most $\aleph_1$ many predense subsets of $\mathbb{P}$, each of cardinality at most $\aleph_1$, then there is a $\mathscr{D}$-generic filter $\mathscr{G} \subseteq \mathbb{P}$.
2. BSPFA^{++} holds iff, with $\mathbb{P}, \mathscr{D}$ as above, if in addition a sequence
$$\langle \tau_\alpha : \alpha < \omega_1 \rangle$$
of $\mathbb{P}$-names for stationary subsets of ω_1 is given, then there is a $\mathscr{D}$-generic filter $\mathscr{G} \subseteq \mathbb{P}$ such that for all $\alpha < \omega_1$,
$$(\tau_\alpha)_{\mathscr{G}} := \{\beta < \omega_1 : \exists p \in \mathscr{G}(p \Vdash \beta \in \tau_\alpha)\}$$
is stationary in ω_1.

Definition 1.5 (Woodin [32, Definition 5.12]). Let S and T be stationary, co-stationary subsets of ω_1. $\psi_{\mathsf{AC}}(S, T)$ is the following statement: Let NS_{ω_1} be the nonstationary ideal on ω_1. Let $\mathbb{P} = \mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. Then there is an $\alpha < \omega_2$ such that whenever G is a $\mathbb{P}$-generic filter over V, then
$$S \in G \quad \text{iff} \quad \alpha \in j(T),$$
where $j : V \to (\mathrm{Ult}(V, G), \tilde{\in}) \subseteq V[G]$ is the generic ultrapower embedding (and, as customary, we identify the standard part of a model with its transitive collapse.)

ψ_{AC} is the statement that $\psi_{\mathsf{AC}}(S, T)$ holds for any S and T stationary, co-stationary subsets of ω_1.

Definition 1.5 can be restated without mentioning the generic: Given S and T as above, the condition on α is equivalent to
$$[S]_{\mathrm{NS}_{\omega_1}} = [\![\alpha \in \dot{j}(T)]\!]_{\mathrm{RO}(\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1})},$$
where $\dot{j}$ is with Boolean value $\mathtt{1}$ a $\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$-name for the generic ultrapower embedding.

In turn, this is equivalent to stating the existence of a bijection $\pi : \omega_1 \to \alpha$ and of a club $C \subseteq \omega_1$ such that
$$S \cap C = \{\beta \in C : \mathrm{ot}(\pi``\beta) \in T\}.$$

Theorem 1.6. *Let $L[\mathscr{E}]$ be a fine structural inner model with a strong cardinal but without inner models with Woodin cardinals. Then there is a forcing extension of $L[\mathscr{E}]$ where the following hold*:

1. $\mathsf{SPFA}(\mathfrak{c}) + \mathsf{BSPFA}^{++}$.
2. ψ_{AC}.
3. *All $\utilde{\Delta}^1_3$-sets of reals are Lebesgue measurable and have the property of Baire.*
4. *There is a projective lightface (in fact, Σ^1_6) well-ordering of $\mathbb{R}$.*

Remark 1.7.

1. Since ψ_{AC} (as well as BPFA, the bounded *proper* forcing axiom) implies $\mathfrak{c} = \aleph_2$ (see [32, Lemma 5.15]), it follows that MA holds in the extension described in the theorem.
2. That a strong cardinal suffices to obtain $\mathsf{SPFA}(\mathfrak{c})$ by forcing was known before, see [32, Remark 2.48]. Originally, this was somewhat surprising, given the equivalence between SPFA and MM. Woodin was first to show that $\mathsf{SPFA}(\mathfrak{c})$ is strictly weaker than $\mathsf{MM}(\mathfrak{c})$. For example, [32, Theorem 9.73] states that

$\mathsf{MM}(\mathfrak{c})$ implies Projective Determinacy and is therefore in consistency strength strictly above $\mathsf{SPFA}(\mathfrak{c})$.

3. In fact, a strong cardinal is much more than necessary, and we just use it here to speed up the argument. However, notice that if there is a strong cardinal, then all sets have sharps, and we have the following result (see [23] or [6, Theorem 3]).

 THEOREM 1.8 (Martin, Solovay). *If all sets have sharps, then Σ^1_3-absoluteness holds, i.e., for every Σ^1_3-formula $\varphi(\vec{x})$, any two-step forcing notion $\mathbb{P} * \dot{\mathbb{Q}}$, and any $\vec{r} \in \mathbb{R}^{V^{\mathbb{P}}}$,*

$$V^{\mathbb{P}} \models \varphi(\vec{r}) \Leftrightarrow V^{\mathbb{P}*\dot{\mathbb{Q}}} \models \varphi(\check{\vec{r}}). \qquad \square$$

4. A well-known forcing axiom of a different kind than the ones we have stated above is the *Open Coloring Axiom* OCA. See [11] for a thorough treatment of this axiom. OCA was defined by Todorcevic, see [31, Theorem 8.0 and subsequent comments]. It holds in the model of Theorem 1.6 since it is a consequence of $\mathsf{SPFA}(\mathfrak{c})$, see for example [11, Proposition 43].
5. For Γ a (boldface) pointclass let $\Gamma(\mathscr{L})$ and $\Gamma(\mathscr{M})$ denote respectively the statements that Γ-sets of reals are Lebesgue measurable and that they have the property of Baire. Recall that Martin and Solovay [24, §4] proved that MA implies $\boldsymbol{\Sigma}^1_2(\mathscr{L})$ and $\boldsymbol{\Sigma}^1_2(\mathscr{M})$; this also follows from the existence of sharps for reals (see for example [4, Corollary 9.3.2]). Judah has shown (see for example [19, Theorem 4.4]) that MA implies neither $\boldsymbol{\Delta}^1_3(\mathscr{L})$ nor $\boldsymbol{\Delta}^1_3(\mathscr{M})$.
6. It is a folklore result that no well-ordering of the reals is Lebesgue measurable, see [5, Theorem 1.2]. Since any $\boldsymbol{\Sigma}^1_3$-total ordering is $\boldsymbol{\Delta}^1_3$, it follows that the complexity of the well-ordering cannot be improved to be $\boldsymbol{\Sigma}^1_3$. It is open whether the complexity of the well-ordering can be improved to be $\boldsymbol{\Sigma}^1_4$ or even $\boldsymbol{\Sigma}^1_5$.

We are appreciative of the referee's useful comments, which helped us to present the proof in a more clear and detailed manner. Thanks are also due to Benjamin Miller for stylistic comments. Of course, all inaccuracies and mistakes that remain are entirely the author's fault.

§2. Fine structural prolegomena. Our fine structural terminology is standard, we follow the Mitchell-Steel style. As requested by the referee, we have included some references and explanations, we isolate them in this section to avoid breaking the flow of the proof in §3. Due to the technical nature of the notions involved, these explanations are unfortunately rather vague, and any interested reader should consult the works cited for precise statements and definitions. The basic reference for fine structure is Jensen's seminal work [18]; for a modern and streamlined presentation we suggest [29]. Fine structural models are defined inductively by stages, much like L is defined. Given a predicate A, the stages in the construction of $L[A]$ are denoted by J^A_α, these are transitive structures, $\mathrm{ORD} \cap J^A_\alpha = \omega\alpha$, $J^A_\alpha \subset J^A_\beta$ for $\alpha < \beta$, and they resemble closely the L-hierarchy in that $J^A_{\alpha+1} \cap \mathscr{P}(J^A_\alpha) = \mathrm{Def}(J^A_\alpha)$; they are better behaved in the sense that they are closed under elementary operations like forming ordered pairs, see [18, §2] or [29, §1].

Mitchell-Steel models $L[\mathscr{E}]$ are usually called *weasels*; the set models that serve as stages of the construction of a weasel are usually called *mice*. A few preliminary notions are required before we can state what a mouse is. The first such notion is that of *potential premouse*, it is introduced and defined in [26, Definition 1.0.5], which builds on the notion of *good extender sequence* introduced in [26, Definition 1.0.4]; unfortunately, this definition is incomplete. The definition is completed in [28, Definition 2.4], or see the excellent [30] where the notion is renamed *fine extender sequence* in [30, Definition 2.4]. A transitive structure $\mathscr{M}$ is a *potential premouse (ppm)* iff it is a structure of the form $(J^{\mathscr{E}}_\alpha, \in, \mathscr{E}\restriction_\alpha, \mathscr{E}_\alpha)$ where $\mathscr{E}$ is a fine extender sequence, see [26, Definition 1.0.5], [30, Definition 2.6]. It is customary to denote $\mathscr{M}$ by $\mathscr{J}^{\mathscr{E}}_\alpha$. For $\beta \leq \alpha$ it is also customary to denote $\mathscr{J}^{\mathscr{E}}_\beta$ by $\mathscr{J}^{\mathscr{M}}_\beta$. Without going into details, a *fine extender sequence* is a sequence $\mathscr{E}$ such that for each α in its domain, $\mathscr{E}_\alpha = \emptyset$ or else $\mathscr{E}_\alpha$ is a (κ, α)-extender over $J^{\mathscr{E}}_\alpha$ for some $\kappa \in J^{\mathscr{E}}_\alpha$.

Two basic fine structural objects related to a potential premouse $\mathscr{M}$ are its *first standard parameter* $p_1(\mathscr{M})$ and its Σ_1-*projectum* $\rho_1(\mathscr{M})$, see [30, Definitions 2.12 and 2.13]. Potential premice $\mathscr{M} = (J^{\mathscr{E}}_\alpha, \in, \mathscr{E}\restriction_\alpha, \mathscr{E}_\alpha)$ can be nicely coded by their Σ_0-*code* $\mathscr{C}_0(\mathscr{M})$. We will not define this structure here, but we point out that it has the form

$$\left(J^{\mathscr{E}}_{\beta^{\mathscr{M}}}, \in, \mathscr{E}\restriction_{\beta^{\mathscr{M}}}, \mathscr{E}^*_{\beta^{\mathscr{M}}}, \dot{\mu}^{\mathscr{M}}, \dot{\nu}^{\mathscr{M}}, \dot{\gamma}^{\mathscr{M}}\right)$$

for appropriately defined predicates $\dot{\mu}^{\mathscr{M}}, \dot{\nu}^{\mathscr{M}}, \dot{\gamma}^{\mathscr{M}}$ and ordinal $\beta^{\mathscr{M}}$, and where $\mathscr{E}^*_{\beta^{\mathscr{M}}}$ is a “nice” predicate coding $\mathscr{E}_{\beta^{\mathscr{M}}}$, see [30, Definition 2.11]. $\mathscr{C}_0(\mathscr{M})$ is amenable, which makes it better suited for fine structural analysis than $\mathscr{M}$ itself. It is basically harmless to identify $\mathscr{M}$ with its Σ_0-code, and we have done so in some of the (sketched) definitions in the paragraphs below. The Σ_1-*projectum* of $\mathscr{M}$ is the least ordinal α such that there is a set $A \subseteq \alpha$ such that $A \notin \mathscr{C}_0(\mathscr{M})$ but A is Σ_1-definable over $\mathscr{C}_0(\mathscr{M})$ with parameters. These parameters can be taken to be decreasing sequences of ordinals, so they are well-ordered lexicographically. The least parameter from which such a set A as above can be defined is the *first standard parameter* of $\mathscr{M}$.

The *first core* of $\mathscr{M}$ is the structure

$$\mathscr{C}_1(\mathscr{M}) = \mathscr{H}_1^{\mathscr{C}_0(\mathscr{M})}\left(\rho_1(\mathscr{M}) \cup \{p_1(\mathscr{M})\}\right),$$

where $\mathscr{H}_1$ denotes the transitive collapse of the corresponding Σ_1-hull, see [30, Definition 2.14]. Using these notions the concepts of *1-solidity* and *1-soundness* of $\mathscr{M}$ can be defined (for example $\mathscr{M}$ is 1-sound iff it is 1-solid and $\mathscr{C}_1(\mathscr{M}) = \mathscr{C}_0(\mathscr{M})$), see [30, Definitions 2.15 and 2.16]. We will not explain here what solidity means. By a delicate induction that occupies a significant part of [26], the concepts of *n-solid* and *n-sound* potential premice can then be defined, see [26, Definition 2.8.3], [26, §8] and [28, §3]. Very roughly speaking, soundness of $\mathscr{M} = \mathscr{J}^{\mathscr{E}}_\alpha$ entails the following: Suppose that a subset X of an ordinal κ belongs to $\mathscr{J}^{\mathscr{M}}_{\beta+1} \setminus \mathscr{J}^{\mathscr{M}}_\beta$. Then there is a surjection $f \vdots \kappa \to \mathscr{J}^{\mathscr{M}}_\beta$ in $\mathscr{J}^{\mathscr{M}}_{\beta+1}$. Moreover, for some m, X must be Σ_m-definable over $\mathscr{J}^{\mathscr{M}}_\beta$ from parameters in $\mathscr{J}^{\mathscr{M}}_\beta$; if $m \leq n$, then such a surjection f can be chosen to be Σ_m-definable as well.

A potential premouse is *sound* or ω-*sound* iff it is n-sound for all $n < \omega$, see [30, Definition 2.17]. In Mitchell-Steel models $\mathscr{M}$, all levels $\mathscr{J}^{\mathscr{M}}_\alpha$ are ω-sound. This implies that fully elementary Skolem hulls of $\mathscr{M}$ are ω-sound as well. Structures that are not ω-sound can appear, however, when forming ultrapowers, see [26, §4], especially [26, Lemmas 4.5 and 4.6]; a remark indicating how soundness fails is unfortunately incomplete in [26], for this see [30, Lemma 2.23]. It is not hard to see that if $\mathscr{M}$ is countable then ω-soundness of $\mathscr{M}$ is an arithmetic statement about x for any real x coding $\mathscr{M}$; it is of course not expressible by a first order sentence in $\mathscr{M}$ itself.

The definitions of soundness and solidity are such that a potential premouse $\mathscr{M}$ needs to be at least $(n-1)$-solid in order to even define its n-projectum and its n-standard parameter, in terms of which n-solidity and n-soundness are then defined. If $\mathscr{M}$ is n-solid for all $n<\omega$ (for example, if $\mathscr{M}$ is ω-sound) then the sequence $(\rho_n(\mathscr{M})\colon n<\omega)$ of projecta of $\mathscr{M}$ is well-defined and non-increasing, so its limit (eventual value) exists. This ordinal is denoted $\rho_\omega^{\mathscr{M}}$, see [30, Definition 2.17].

A *premouse* is a potential premouse all of whose proper initial segments are ω-sound, see [30, Definition 2.19]. Given a potential premouse $\mathscr{M}=\mathscr{J}_\alpha^{\mathscr{E}}$, a potential premouse $\mathscr{N}$ is an *initial segment* of $\mathscr{M}$ iff $\mathscr{N}=\mathscr{J}_\beta^{\mathscr{M}}$ for some $\beta\leq\alpha$. If $\beta<\alpha$ we say that $\mathscr{N}$ is a *proper* initial segment of $\mathscr{M}$, see [30, Definition 2.18].

A *mouse* is an iterable premouse $\mathscr{J}_\alpha^{\mathscr{E}}$. *Iterability* is a delicate notion, closely related to the large cardinal strength the sequence $\mathscr{E}$ codifies. If $\mathscr{J}_\alpha^{\mathscr{E}}$ is countable, the most generous iterability notion defined in [30] is $(\omega,\omega_1,\omega_1+1)$-iterability, see [30, Definition 4.3]; this is a Σ_3^2 notion, far too complicated for our present purposes. In the absence of Woodin cardinals, there is a Π_2^1 notion of iterability that suffices in the sense that iterable premice are comparable, see for example [25], the introduction to which we strongly recommend. For a general definition of *comparison* see for example [30, Theorem 3.11]; we say that two sufficiently iterable premice $\mathscr{M}$ and $\mathscr{N}$ are *comparable* iff there are iterations of $\mathscr{M}$ and $\mathscr{N}$ producing final models $\mathscr{M}^\alpha$ and $\mathscr{N}^\beta$ one of which is an initial segment of the other. By *the mouse condition* we refer to the fact that iterability of a countable premouse $\mathscr{M}$ is Π_2^1-definable in any real x coding $\mathscr{M}$. To say that a real x codes a potential premouse $\mathscr{M}$ is itself a Π_1^1 condition, being the conjunction of a first order statement about the model coded by x and the statement that this model is well-founded, see [26, Lemmas 2.5 and 3.3] and [28, Lemmas 2.5 and 2.6 and subsequent remarks]. In Mitchell-Steel models $\mathscr{M}$, all levels $\mathscr{J}_\alpha^{\mathscr{M}}$ are iterable; more generally, if $\mathscr{M}$ is a mouse then all its initial segments are iterable as well.

We close this section by quoting two results from [30] that we will have occasion to use in §3. The first is a very useful tool to show that certain objects are unique, the second is a *condensation* theorem.

THEOREM 2.1. [30, Corollary 3.12] *Let $\mathscr{M}$ and $\mathscr{N}$ be ω-sound mice such that $\rho_\omega^{\mathscr{M}}=\rho_\omega^{\mathscr{N}}=\omega$. Then $\mathscr{M}$ is an initial segment of $\mathscr{N}$ or vice versa.* □

THEOREM 2.2. [30, Theorem 5.1] *Let $\mathscr{M}$ be an ω-sound mouse. Suppose that $\pi\colon\mathscr{N}\to\mathscr{M}$ is elementary and* $\mathrm{cp}(\pi)=\rho_\omega^{\mathscr{N}}$. *Then either*

1. *$\mathscr{N}$ is a proper initial segment of $\mathscr{M}$, or*
2. *There is an extender E on the extender sequence of $\mathscr{M}$ such that E has length $\rho_\omega^{\mathscr{N}}$ and $\mathscr{N}$ is a proper initial segment of* $\mathrm{Ult}_0(\mathscr{M},E)$. □

In the statement of Theorem 2.2, Ult_0 denotes the standard internal ultrapower, see [30, §2.4].

§3. The proof. This section is devoted to the proof of Theorem 1.6. The proof divides in a natural way into four parts: First we define for κ a strong cardinal, a revised countable support iteration $\mathbb{P}_\kappa$ of length κ of semiproper forcings, and show that

$$V^{\mathbb{P}_\kappa} \models \mathsf{SPFA}(\mathfrak{c}) + \mathsf{BSPFA}^{++}.$$

Second, for any measurable cardinal ξ and any S and T stationary, co-stationary subsets of ω_1, we define a semiproper forcing $\mathbb{Q}$ of size ξ that forces $\psi_{\mathsf{AC}}(S, T)$ to hold. Notice that once it holds, $\psi_{\mathsf{AC}}(S, T)$ is preserved by any further ω_1-preserving extensions—or stationary set preserving, if we want S and T to stay stationary, co-stationary in the extension. We use this to argue that the construction of part one satisfies

$$V^{\mathbb{P}_\kappa} \models \psi_{\mathsf{AC}}.$$

This elaborates on an argument of Woodin [32, Lemma 10.95] that we also improve (since the forcing axiom we are assuming is strictly weaker than BMM.) From ψ_{AC} it follows that a well-ordering of $\mathbb{R}$ can be easily defined from an infinite sequence of pairwise disjoint stationary sets.

Third, we use Σ^1_3-absoluteness and results of Judah [19] to argue that

$$V^{\mathbb{P}_\kappa} \models \underset{\sim}{\mathbf{\Delta}}^1_3(\mathscr{L}) \text{ and } \underset{\sim}{\mathbf{\Delta}}^1_3(\mathscr{M}).$$

Finally, suppose $L[\mathscr{E}]$ is as in the hypothesis of the theorem, λ is strong in $L[\mathscr{E}]$ and $\mathbb{P} = (\mathbb{P}_\lambda)^{L[\mathscr{E}]}$. Then

$$L[\mathscr{E}] \models \text{There is a } \Delta^1_3\text{-in-the-codes sequence } (S_n : n < \omega) \text{ of disjoint stationary subsets of } \omega_1.$$

By Σ^1_3-absoluteness between $L[\mathscr{E}]$ and $L[\mathscr{E}]^{\mathbb{P}}$, such a sequence is still Δ^1_3-definable in $L[\mathscr{E}]^{\mathbb{P}}$, and by definition of $\mathbb{P}$ the S_n are still stationary in ω_1. A calculation then shows that

$$L[\mathscr{E}]^{\mathbb{P}} \models \text{There is a } \Sigma^1_6\text{-well-ordering of the reals,}$$

completing the proof.

Remark 3.1. That BSPFA is strictly weaker than BMM is the content of [1, Theorem 3.5]. In [1, Corollary 2.3] it is shown that the statement of [32, Lemma 10.95] can be improved by replacing the measurable with a cardinal κ satisfying the Erdős property $\kappa \to (<\omega_1)^{<\omega}_{2^{\omega_1}}$, see [1] for the relevant definitions. Schindler [27, Theorem 1.1] shows that even in consistency strength BMM is significantly stronger than BSPFA. Namely, BSPFA is equiconsistent with the existence of Σ_1-reflecting cardinals, while BMM implies that every set belongs to an inner model with a strong cardinal.

Now we proceed to the details:

For revised countable support (RCS) iterations, the reader is advised to consult [8].

Let κ be strong. The key to defining $\mathbb{P}_\kappa$ is an appropriate version of Laver functions for strong or locally strong cardinals due to Shelah and Gitik, see also [14, Theorem 6].

LEMMA 3.2 (Gitik, Shelah [12, Lemma 2.1]). *Let κ be strong. Then there is $\ell \,\vdots\, \kappa \to V_\kappa$ such that for every x and every $\lambda \geq |\mathrm{TrCl}(x)|$, there is a (κ, λ)-extender E which is λ-strong and such that $j_E(\ell)(\kappa) = x$, where $j_E \colon V \to \mathrm{Ult}(V, E)$ is the ultrapower embedding given by E.* □

Actually, the result in [12] is based on a notion different from that of a (κ, λ)-λ-strong extender: that of a (κ, λ)-normal ultrafilter (as defined in [3, Definition 2.3]) but the proof adapts in a straightforward way. In fact, in the original argument given in [22] the use of supercompactness can be replaced with just strongness. Basically, Laver assumes the result fails, and picks counterexamples for all $f \,\vdots\, \kappa \to V_\kappa$. Let λ_f be a minimal counterexample to such a function f in the sense that λ_f is least such that there is x with $\lambda_f \geq |\mathrm{TrCl}(x)|$, but there is no (κ, λ_f)-λ_f-strong extender E witnessing $j_E(f)(\kappa) = x$. Then consider $j \colon V \to M$, where j comes from a (κ, λ)-λ-strong extender for λ bigger than all the λ_f. Laver uses λ-supercompactness to argue that λ_f is still a counterexample to f in M, from which via a reflection argument a contradiction is easily reached. But all we need is that M contains enough of V, so it sees that there are no (κ, λ_f)-λ_f-strong extenders as required.

We can avoid the argument by contradiction and instead proceed directly, defining ℓ inductively. Such an argument is presented in [14, Theorem 1]. The naïve approach has the disadvantage of requiring a global choice function, but such a predicate can be added by proper class forcing without adding any new sets; the proof in [14, Theorem 1] avoids this step by a reflection argument. See also [14, Theorem 2 and Observation 3] for more on this issue.

The argument works locally, so appropriate Laver functions exist if κ is only θ-strong for some θ, see for example [14, Corollary 7].

Now we can apply the standard proof of the consistency of SPFA, but working with κ strong: An RCS iteration $\langle \mathbb{P}_\alpha \colon \alpha \leq \kappa \rangle$ is defined, so the α^{th} iterand is a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}_\alpha$ for a semiproper forcing such that $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha} \models |\mathbb{P}_\alpha| \leq \aleph_1$, and if $\ell(\alpha) = (\dot{\mathbb{Q}}, \mathscr{D})$, where

- ℓ is our Laver function,
- $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name for a semiproper forcing such that $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}} \models |\mathbb{P}_\alpha| \leq \aleph_1$, and
- $\mathscr{D}$ is a $\mathbb{P}_\alpha$-name for a sequence of $\gamma < \kappa$ dense subsets of $\dot{\mathbb{Q}}$,

then $\dot{\mathbb{Q}}_\alpha$ is defined as $\dot{\mathbb{Q}}$.

Forcing with $\mathbb{P}_\kappa$ is semiproper, see [8, Theorem 4.1], so ω_1 and its stationary subsets are preserved. It is κ-cc, because at inaccessible points direct limits are taken in RCS iterations, see [16, Theorem II.7.9].

LEMMA 3.3. $V^{\mathbb{P}_\kappa} \models \mathsf{SPFA}(\mathfrak{c})$.

PROOF. The usual proof of the consistency of PFA adapts (see [16, Theorem III.6.7], [9, §1] or [8, Theorem 5.1]). Clearly, $V^{\mathbb{P}_\kappa} \models \kappa = \mathfrak{c} = \aleph_2$. Let G be $\mathbb{P}_\kappa$-generic over V. In $V[G]$, let $\mathbb{Q}$ be semiproper, $|\mathbb{Q}| \leq \mathfrak{c}$, and let $\mathscr{D}$ be a sequence of length $< \mathfrak{c}$ of dense subsets of $\mathbb{Q}$. We must show that in $V[G]$ there is a $\mathscr{D}$-generic filter $\mathscr{G} \subseteq \mathbb{Q}$.

Let $j \colon V \to M$ be a witness to the 3-strength of κ, i.e., M is transitive, $\mathrm{cp}(j) = \kappa$ and $V_{\kappa+3} \subset M$. Then $\mathbb{P}_\kappa \in M$ and $\mathbb{Q}, \mathscr{D} \in M[G]$ (as names for them can be coded by elements of $V_{\kappa+2}$). Choose j so, in addition,

$$j(\ell)(\kappa) = (\mathbb{P}_\kappa\text{-name for } \mathbb{Q} * \dot{\mathtt{Coll}}(\omega_1, |\mathbb{Q}|^+), \mathbb{P}_\kappa\text{-name for } \tilde{\mathscr{D}})$$

where letting $A = \texttt{Coll}(\dot{\omega_1}, |\mathbb{Q}|^+)$, if $\mathscr{D} = (D_\alpha \colon \alpha < \gamma)$ we set $\tilde{\dot{\mathscr{D}}} = (D_\alpha \times A\colon \alpha < \gamma)$. Notice that $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{S}}$ for some name for a poset $\dot{\mathbb{S}}$, both in M and in V.

We claim that $M[G] \models \mathbb{Q}$ is semiproper. To see this, we use the characterization of semiproperness in terms of games (see [16, Definition III.7.7]):

> Let $\mathbb{Q}$ be a forcing. Consider the following game of length ω between players I and II, with I moving first: Player I plays a condition in $\mathbb{Q}$, and then they alternate, I playing $\mathbb{Q}$-names for countable ordinals, and II playing countable ordinals. II wins iff some condition extending the one I played forces each name to be one of the ordinals played.
>
> $\mathbb{Q}$ is semiproper iff II has a winning strategy.

Since $\mathbb{Q}$ is semiproper in $V[G]$, II has a winning strategy in $V[G]$. Formally, this is a definable class but it suffices to think of it as a function with domain, say, nice names for countable ordinals. A nice name for a countable ordinal is defined from a sequence of antichains. Since $V_{\kappa+2} \subset M$, both $M[G]$ and $V[G]$ agree on what the nice names for countable ordinals are, and since in fact $V_{\kappa+3} \subset M$, we can ensure that II's winning strategy belongs to $M[G]$, since it can be coded as a subset of the set of nice names for countable ordinals, a name for which can be in turn coded in $V_{\kappa+3}$. The claim follows, and from the claim we see that in M, $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \texttt{Coll}(\dot{\omega_1}, |\mathbb{Q}|^+) * \dot{\mathbb{T}}$ for some $\dot{\mathbb{T}}$ and therefore in V, $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{W}}$ for some $\dot{\mathbb{W}}$.

Let H be a $j(\mathbb{P}_\kappa)$-generic over V extending G, so $H = G * G' * G''$ where G' is $\mathbb{Q}$-generic over $V[G]$. Since $\mathbb{P}_\kappa$ is κ-cc, j lifts in $V[H]$ to $\hat{j}\colon V[G] \to M[H]$. Since G' is essentially a subset of κ, we can without loss assume that $\hat{j}``G' = G'$. But since $V_{\kappa+2} \subset M$, $G' \in M[G]$. Clearly, $\hat{j}``G'$ meets each $j(D_\alpha)$, $\alpha < \gamma$, and since $\gamma < \kappa$, $\hat{j}(\mathscr{D}) = \hat{j}``\mathscr{D}$, so the filter generated by G' in $j(\mathbb{Q})$ is $\hat{j}(\mathscr{D})$-generic. It follows by elementarity that in $V[G]$ there is a $\mathscr{D}$-generic filter. □

REMARK 3.4. The argument given shows that strength is more than is actually needed. The function ℓ only needs to predict small objects, and it follows from the proof that 3-strength of κ suffices.

Notice that the argument just given is soft enough that allows for additional clauses, thus providing a method for showing the consistency of SPFA($\mathfrak{c}$) together with several other principles. For example, these clauses can be used to implement BSPFA by copying the argument in the proof of [13, Theorem 2.11], replacing "countable support iteration" with "RCS iteration" and "proper" with "semiproper". Since strong cardinals are Σ_1-reflecting, this argument works. That in fact BSPFA^{++} holds in the extension can be ensured in a straightforward way by adding an additional clause to requirements (1)–(6) in the proof of [13, Theorem 2.11]. So, together with the iteration and the sequences of names $\underset{\sim}{\mathfrak{M}}_i$ as defined there, we also have a list of names for ω_1-sequences of names for stationary sets, and stipulate that they are met. We can order our list so each name appears stationarily often, and this guarantees the sufficiently generic filters will guess them as desired. This shows the required result:

LEMMA 3.5. $V^{\mathbb{P}_\kappa} \models$ BSPFA^{++}. □

This concludes the first part of the proof. For the second one, we start with a diversion:

FACT 3.6. *Suppose that* BSPFA *holds and that there is a measurable cardinal. Then* ψ_{AC} *holds.*

PROOF. This is like [32, Lemma 10.95]. Let S and T be stationary, costationary subsets of ω_1. We only need to verify the following: Let ξ be a measurable cardinal and let $\mathbb{Q}$ be the forcing that collapses ξ to ω_1 via a bijection $\pi\colon \omega_1 \to \xi$ while shooting a club $C \subseteq \omega_1$ such that

$$T \cap C = \{\alpha \in C : \pi``\alpha \in S\}.$$

Conditions in $\mathbb{Q}$ are closed initial segments of the intended bijection, and the order is by extension. Then $\mathbb{Q}$ is semiproper (in [32, Lemma 10.95] the weaker claim is made that it is stationary set preserving.)

In effect, let $\eta > \xi$ be sufficiently large and let $X \prec V_\eta$ be countable and contain all relevant parameters. We can assume $X \cap \omega_1 \in T$. Since ξ is measurable, X can be expanded to a structure Y such that $Y \cap \omega_1 = X \cap \omega_1$ yet $\mathrm{ot}(Y \cap \xi) \in S$. This can be easily achieved by standard arguments. For example, by iterating the construction in [20, Lemma 1.1.21].

With Y as above, if p is the union of a Y-generic chain of conditions (i.e., a descending ω-sequence of conditions in Y meeting every dense set in Y), then $p \cup \{(Y \cap \omega_1, \mathrm{ot}(Y \cap \xi))\}$ *is* a condition in $\mathbb{Q}$ which is clearly X-generic, and semiproperness follows. By our forcing axiom, $\psi_{\mathsf{AC}}(S,T)$ must hold. Since S and T are arbitrary, we are done. □

Similarly,

LEMMA 3.7. *With* κ *a strong cardinal and* $\mathbb{P}_\kappa$ *as above,* $V^{\mathbb{P}_\kappa} \models \psi_{\mathsf{AC}}$.

PROOF. Just notice that the measurable cardinals are cofinal in κ and that given S and T, stationarily often below κ the Laver function ℓ will predict a forcing like the one described in Fact 3.6 to be used along the inductive construction of $\mathbb{P}_\kappa$. An easy bookkeeping gives now the result. □

REMARK 3.8. Before proceeding with the argument, it is worth pointing out that a variation of the proof of Fact 3.6 allows us to give an easy forcing construction of a model of club bounding starting with the optimal hypothesis. Recall that *club bounding* is the statement that every function $f : \omega_1 \to \omega_1$ is dominated on a club by a canonical function (i.e., a function $h\colon \omega_1 \to \omega_1$ such that for some $\alpha < \omega_2$ and some bijection $g\colon \omega_1 \to \alpha$, for all $\beta < \omega_1$, $h(\beta) = \mathrm{ot}(g``\beta)$.) It is a consequence of, say, [21, Corollary 5.2], that an inaccessible limit of measurable cardinals suffices to force a model of club bounding; even though [21] is probably the first printed reference where this result appears, it is probably folklore and was known long before [21], where a strengthening (its consistency with CH) is obtained. That an inaccessible limit of measurable cardinals is precisely its consistency strength is proved in [7, Theorem 1].

To see that an inaccessible limit of measurable cardinals suffices, notice that given $f : \omega_1 \to \omega_1$ and a measurable κ, an easy variation of the proof above shows that the natural forcing adding a canonical function dominating f (with index κ) is semiproper. By easy bookkeeping, if λ is regular and limit of measurable cardinals, an RCS iteration of length λ of these forcings produces a model as desired.

Since we will need this explicit definition, let us now recall how ψ_{AC} can be used to provide us with well-orderings of $\mathbb{R}$.

Suppose $(S_n : n < \omega)$ is a sequence of disjoint stationary subsets of ω_1. We associate to each $x \subseteq \omega$ the set $S_x = \bigcup\{S_{i+1} : i \in x\}$. Notice S_x is stationary, co-stationary. The ordinal γ_x is defined from S_x as the least γ such that

$$[S_x]_{\mathrm{NS}_{\omega_1}} = [\![\gamma \in j(S_0)]\!]_{\mathrm{RO}(\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1})}$$

where j is as before. That γ_x exists is precisely what $\psi_{\mathsf{AC}}(S_x, S_0)$ asserts.

Notice that if x and y are distinct subsets of ω, then $\gamma_x \neq \gamma_y$, as $S_x \neq_{\mathrm{NS}_{\omega_1}} S_y$. This completes the second part of the proof.

Now we proceed to establish $\utilde{\mathbf{\Delta}}^1_3(\mathscr{L})$ and $\utilde{\mathbf{\Delta}}^1_3(\mathscr{M})$. Fix Σ^1_3-formulas φ, ψ with real parameters such that

$$V^{\mathbb{P}_\kappa} \models \forall x \in \mathbb{R}\big(\varphi(x) \leftrightarrow \neg\psi(x)\big).$$

Since κ is weakly compact and $\mathbb{P}_\kappa$ is κ-cc, it follows by a well-known argument that every real in $V^{\mathbb{P}_\kappa}$ must appear at an intermediate extension $V^{\mathbb{P}_\alpha}$, $\alpha < \kappa$. We may thus assume that the parameters in φ and ψ are in V. By Σ^1_3-absoluteness, $V \models \forall x \in \mathbb{R}\big(\varphi(x) \leftrightarrow \neg\psi(x)\big)$, and the same holds for any intermediate model between V and $V^{\mathbb{P}_\kappa}$. In any such model, the statement "$\{x \in \mathbb{R} : \varphi(x)\}$ is Lebesgue measurable" is $\utilde{\mathbf{\Sigma}}^1_4$ as it is equivalent to

$$\text{“}\exists B \ \mathsf{Borel}\big(\mu\big(\{x \in \mathbb{R} : \big(x \in B \wedge \psi(x)\big) \vee \big(\varphi(x) \wedge x \notin B\big)\}\big) = 0\big)\text{”}$$

where μ is Lebesgue measure, and for $\utilde{\mathbf{\Sigma}}^1_n$-sets A, the statement "$\mu(A) = 0$" is $\utilde{\mathbf{\Sigma}}^1_{n+1}$, see for example [4, Lemma 9.1.2]. Similarly, the statement "$\{x \in \mathbb{R} : \varphi(x)\}$ has the Property of Baire" is $\utilde{\mathbf{\Sigma}}^1_4$ in V and in any $V^{\mathbb{P}_\alpha}$, $0 \leq \alpha \leq \kappa$.

By [19, Corollary 1.9], if $\utilde{\mathbf{\Sigma}}^1_2(\mathscr{L})$ then

$$1 \Vdash_{\mathtt{Random}_{\omega_1}} \utilde{\mathbf{\Delta}}^1_3(\mathscr{L}),$$

where $\mathtt{Random}_\zeta$ is the standard forcing notion for adding ζ many random reals, and

$$1 \Vdash_{\mathtt{Add}(\omega,\omega_1)} \utilde{\mathbf{\Delta}}^1_3(\mathscr{M}),$$

where $\mathtt{Add}(\omega, \zeta)$ is the standard forcing notion for adding ζ many Cohen reals. Now notice that for some, in fact for cofinally many $\alpha < \kappa$, $V^{\mathbb{P}_\alpha} \models \mathbb{Q}_\alpha \cong \mathtt{Random}_{\omega_1} * \dot{\mathbb{S}}$ for some (name for a) semiproper forcing $\dot{\mathbb{S}}$ in $V' = V^{\mathbb{P}_\alpha * \mathtt{Random}_{\omega_1}}$. Since κ is still measurable in $V^{\mathbb{P}_\alpha}$, all reals in $V^{\mathbb{P}_\alpha}$ have sharps and therefore $V^{\mathbb{P}_\alpha} \models \utilde{\mathbf{\Sigma}}^1_2(\mathscr{L})$. It follows that $V' \models$"$\{x \in \mathbb{R} : \varphi(x)\}$ is Lebesgue measurable" and by Σ^1_3-absoluteness, the same holds in $V^{\mathbb{P}_\kappa}$.

The argument for the Baire property is the same, replacing $\mathtt{Random}_{\omega_1}$ with $\mathtt{Add}(\omega, \omega_1)$ and $\utilde{\mathbf{\Sigma}}^1_2(\mathscr{L})$ with $\utilde{\mathbf{\Sigma}}^1_2(\mathscr{M})$.

Since φ, ψ and their parameters were arbitrary, $\utilde{\mathbf{\Delta}}^1_3(\mathscr{L})$ and $\utilde{\mathbf{\Delta}}^1_3(\mathscr{M})$ in $V^{\mathbb{P}_\kappa}$ follow, and this completes the third part of the proof.

Now let $L[\mathscr{E}]$ be as in the hypothesis of Theorem 1.6. We proceed to define in $L[\mathscr{E}]$ a Δ^1_3-in-the-codes ω-sequence of stationary subsets of ω_1. We show that this sequence is still Δ^1_3-definable in $L[\mathscr{E}]^{\mathbb{P}}$, and argue that a Σ^1_6-well-ordering of $\mathbb{R}$ can be obtained from it.

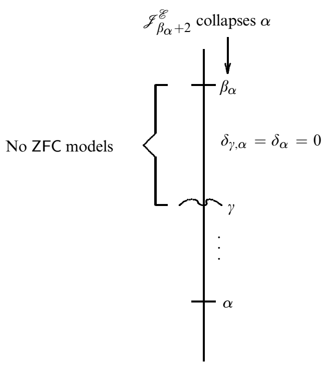

FIGURE 1. Defining S_1.

First we define in $L[\mathscr{E}]$ the sequence $(S_n : n < \omega)$. We verify they are stationary in ω_1. Since $\mathbb{P}$ is an RCS iteration of semiproper forcings, the stationarity of the sets S_n is preserved when forcing with $\mathbb{P}$.

Club many $\alpha < \omega_1$ are a local ω_1, $\alpha = \omega_1^{\mathcal{J}^{\mathscr{E}}_\delta}$, where $\mathcal{J}^{\mathscr{E}}_\delta \models \mathsf{ZFC}$. This club exists, by reflection, since (say) there are inaccessibles.

For such an ordinal α, define β_α as the least ordinal such that $\mathcal{J}^{\mathscr{E}}_{\beta_\alpha+2} \models \alpha$ is countable. Obviously, $\beta_\alpha > \alpha$. For $\gamma \in (\alpha, \beta_\alpha)$, let $\delta_{\gamma,\alpha}$ denote the order type of

$$\{\mathcal{J}^{\mathscr{E}}_\delta : \gamma < \delta < \beta_\alpha,\ \mathcal{J}^{\mathscr{E}}_\delta \models \mathsf{ZFC}\}.$$

Finally, let $\delta_\alpha = \lim_{\gamma \nearrow \beta_\alpha} \delta_{\gamma,\alpha}$. Since $\delta_{\gamma,\alpha}$ decreases as γ increases, δ_α exists.

For all α, $\delta_\alpha = 0$ or else it is an additively closed limit ordinal. If $\delta_\alpha = 0$, set $\alpha \in S_1$. If $\delta_\alpha = \omega^n$ where $0 < n < \omega$, set $\alpha \in S_{n+1}$. Otherwise, set $\alpha \in S_0$.

CLAIM 3.9. *Each S_n is stationary.*

PROOF. Suppose S_n is not. Let C be the first club in the order of definability avoiding S_n. Let κ be least such that, setting $\mathscr{M} = \mathcal{J}^{\mathscr{E}}_\kappa$, then

- $\mathscr{M} \models \mathsf{ZFC}$,
- $C \in \mathscr{M}$,
- Let $\tau = \mathrm{ot}\{\beta < \omega\kappa : \mathcal{J}^{\mathscr{M}}_\beta \models \mathsf{ZFC}$ and $C \in \mathcal{J}^{\mathscr{M}}_\beta\}$. Then either
  - $\tau = 0$ and $n = 1$, or
  - $\tau = \omega^m$, $0 < m < \omega$, and $n = m + 1$, or else
  - $\tau \geq \omega^\omega$ and $n = 0$.

Let $X = \mathtt{Hull}^{\mathscr{M}}(\emptyset)$, so $X \prec \mathscr{M}$, $S_n \in X$ and since κ was chosen so $C \in \mathscr{M}$, then $C \in X$. Let N_X be the transitive collapse of X. Then, by [26, Corollary 2.6], $N_X = \mathcal{J}^{\mathscr{E}'}_\beta$ for some countable ordinal β and fine extender sequence $\mathscr{E}'$, $N_X \models \mathsf{ZFC}$ and N_X is pointwise definable without parameters in $L[\mathscr{E}]$, although we make no use of this last remark in this proof. Let $\alpha = \omega_1 \cap X = \omega_1^{N_X}$. Then α is countable in $\mathcal{J}^{\mathscr{E}}_{\beta+2}$ and therefore $\beta = \beta_\alpha$.

To see this, notice first that $\rho_\omega^{N_X} = \alpha$. In effect, a new subset of α is definable over N_X by coding the theory of N_X using the club $C^{N_X} = C \cap \alpha$ as a parameter. (That this is a new set follows from the minimality of κ; this is a standard and very old fine structural argument, see [26, §2] for details of how this is done.) This shows that $\rho_\omega^{N_X} \leq \alpha$. Since $\rho_\omega^{N_X}$ is a cardinal of N_X, if strict inequality holds then $\rho_\omega^{N_X} = \omega$, but this is impossible: Consider the elementary embedding $\pi\colon N_X \to \mathscr{M}$ given by the inverse of the transitive collapse of X. We have $\mathrm{cp}(\pi) = \alpha$, so if $\rho_\omega^{N_X} = \omega$ then $\rho_\omega^{\mathscr{M}} = \omega$ as well, which implies that $\mathscr{M}$ and in particular $\omega_1^{\mathscr{M}}$ is countable in $L[\mathscr{E}]$, a contradiction since $\omega_1^{\mathscr{M}} = \sup C = \omega_1^{L[\mathscr{E}]}$. It follows that $\rho_\omega^{N_X} = \alpha$. Since $\mathscr{M}$ is an initial segment of $L[\mathscr{E}]$, it is iterable and ω-sound; this allows us to use the Condensation Theorem 2.2 from which it follows that *either* N_X is a proper initial segment of $\mathscr{M}$, in which case we are done since a new subset of α is definable over N_X and therefore belongs to $\mathscr{J}_{\beta+1}^{\mathscr{M}}$ (namely, the theory of N_X in the parameter C^{N_X}), and from this α is seen to be countable by the next stage of constructibility, by minimality of C, or *else* N_X is a proper initial segment of an ultrapower of $\mathscr{M}$ by an extender E on $\mathscr{E}$ with length α. But then $\mathscr{P}^{\mathscr{M}}(\alpha) \supseteq \mathscr{P}^{\mathrm{Ult}_0(\mathscr{M},E)}(\alpha)$, and the result follows as before.

By the requirement on τ and minimality of κ, it follows that $\alpha \in S_n$. But then we obtain a contradiction because $C \in X$, so $\alpha \in C$. □

It follows from the mouse condition that $(S_n : n < \omega)$ is Δ^1_3-in-the-codes: $\alpha \in S_n$ iff there is a real x coding n and a well-ordering of ω in order-type α, and there is a real y coding a mouse $\mathscr{M}$ such that $\alpha \in \mathscr{M}$ and $\mathscr{M}$ serves as a witness to the membership of α in S_n; i.e., $\mathscr{M} \models \mathsf{ZFC}$, α is countable in $\mathscr{M}$ and $\mathscr{M} \models \alpha \in S_n$. Equivalently, $\alpha \in S_n$ iff for every such x and every y coding such an $\mathscr{M}$, $\mathscr{M}$ certifies this membership.

To see that this definition is correct, suppose that $\mathscr{M}_1$ and $\mathscr{M}_2$ are two mice satisfying ZFC, that $n < \omega$, that α is a countable ordinal in both $\mathscr{M}_1$ and $\mathscr{M}_2$, and that both $\mathscr{M}_1$ and $\mathscr{M}_2$ decide the question of membership of α in S_n, perhaps in different ways. By definition of β_α, $\rho_\omega^{\mathscr{N}_1} = \omega$, where $\mathscr{N}_1$ is one of $\mathscr{J}^{\mathscr{M}_1}_{\beta_\alpha^{\mathscr{M}_1}}$ and $\mathscr{J}^{\mathscr{M}_1}_{\beta_\alpha^{\mathscr{M}_1}+1}$, and similarly $\rho_\omega^{\mathscr{N}_2} = \omega$, where $\mathscr{N}_2$ is one of $\mathscr{J}^{\mathscr{M}_2}_{\beta_\alpha^{\mathscr{M}_2}}$ and $\mathscr{J}^{\mathscr{M}_2}_{\beta_\alpha^{\mathscr{M}_2}+1}$. By Theorem 2.1, one of $\mathscr{N}_1$ and $\mathscr{N}_2$ is an initial segment of the other, and it follows immediately that $\beta_\alpha^{\mathscr{M}_1} = \beta_\alpha^{\mathscr{M}_2}$ and that $\mathscr{M}_1$ and $\mathscr{M}_2$ agree with respect to the membership of α in S_n.

Remark 3.10. The same argument produces in L a partition of complexity Δ^1_2. The complexity increases once $L[\mathscr{E}]$ admits Woodin cardinals.

Claim 3.11. *$(S_n : n < \omega)$ is Δ^1_3-in-the-codes in $L[\mathscr{E}]^{\mathbb{P}}$.*

Proof. This is a consequence of Σ^1_3-absoluteness between the ground model and its forcing extension. As shown above, the sequence is defined by two formulas ψ_0 and ψ_1, where ψ_0 and $\neg\psi_1$ are Σ^1_3.

Absoluteness shows that the formulas still define the same sequence in the extension. For any $\alpha < \omega_1^{L[\mathscr{E}]}$ and any $n < \omega$, there is a mouse in $L[\mathscr{E}]$ witnessing $\alpha \in S_n$ iff there is such a mouse in $L[\mathscr{E}]^{\mathbb{P}}$. Since ω_1 is preserved in the extension, we are done. □

Now we use this sequence and argue from ψ_{AC} that a Δ^1_6-well-ordering of the reals can be defined. The well-ordering is simply

$$x < y \quad \text{iff} \quad \gamma_x < \gamma_y,$$

where $x \mapsto \gamma_x$ is as defined above.

Notice that $\mathbb{R}^{L[\mathscr{E}]}$ has size ω_1 and is a Σ^1_3-set in $L[\mathscr{E}]^{\mathbb{P}}$. This is a consequence of absoluteness, considering a good Δ^1_3-well-ordering of $\mathbb{R}$ in $L[\mathscr{E}]$, see the proof of [32, Theorem 3.28]. By $\mathsf{SPFA}(\mathfrak{c})$, we can talk about subsets of ω_1 using this sequence and almost disjoint forcing: Let $\phi(v)$ be a Σ^1_3-formula describing the good well-ordering of $\mathbb{R}^{L[\mathscr{E}]}$ in the sense that for any real a, $\phi(a)$ states that a codes an injective sequence of reals from $\mathbb{R}^{L[\mathscr{E}]}$, for any $c \in \mathbb{R}^{L[\mathscr{E}]}$ there is a d such that $\phi(d)$ and c is one of the reals in the sequence coded by d, and for any a and b, if $\phi(a)$ and $\phi(b)$ then the sequence coded by a is an initial segment of the one coded by b, or vice versa. Using ϕ we can refer to the α^{th} real of $\mathbb{R}^{L[\mathscr{E}]}$ (for any $\alpha < \omega_1$) in a $\utilde{\Delta}^1_3$-way: c is the α^{th} real of $\mathbb{R}^{L[\mathscr{E}]}$ iff there is a b such that $\phi(b)$, b codes a sequence of length at least $\alpha + 1$, and the α^{th} real it codes is c, iff for all b coding such a sequence and such that $\phi(b)$, c is b's α^{th} real.

Given a real z, we say that z codes $A_z \subseteq \omega_1$ iff

$$A_z = \{\gamma \colon \text{The } \gamma^{\text{th}} \text{ real of } \mathbb{R}^{L[\mathscr{E}]} \text{ is almost disjoint from } z\}.$$

Then A_z is $\Delta^1_3(z)$-in-the-codes, where we identify A_z with the set $\hat{A}_z$ of reals b coding ordinals $\gamma \in A_z$, and using almost disjoint forcing we see that any subset of ω_1 is A_z for some z. By "x codes an ordinal" we mean the usual $\Pi^1_1(x)$ rendering of the statement that (ω, x) is well-ordered, where x is seen as a binary relation by identifying it with a subset of $\omega \times \omega$ in some recursive way. For such a real x, let $|x|$ denote the ordinal it codes. Recall that for x, y coding ordinals, the statements $|x| \leq |y|$, $|x| = |y|$, $|x| \neq |y|$ and $|x| < |y|$ are all $\Delta^1_1(x, y)$. In what follows we will have occasion to identify ω_1 with $\omega_1 \times \omega_1$ or with $\omega \times \omega_1$, $\mathbb{R}$ with $\mathbb{R} \times \mathbb{R}$ or with $\omega \times \mathbb{R}$, etc. All of these identifications are assumed fixed and recursive, and all of them will be denoted by $\langle \cdot, \cdot \rangle$. We hope no confusion results from this abuse of language.

It is straightforward to verify that the statement "A_z is club in ω_1" is $\Pi^1_4(z)$-in-the-codes.

The statement "A_z codes a bijection $\pi \colon \omega_1 \to \gamma$" (for $\omega_1 \leq \gamma < \omega_2$) is $\Pi^1_4(z)$ as well, and it means the following: Identify A_z with a subset of $\omega_1 \times \omega_1$. Similarly, identify reals x with pairs of reals $\langle x_0, x_1 \rangle$. Then A_z codes π as above iff (ω_1, A_z) is well-ordered, which is the conjunction of the following clauses:

- $\forall x \, \forall y \, \forall t \big(x, y, t \text{ code ordinals} \wedge \hat{A}_z\langle x, y\rangle \wedge \hat{A}_z\langle y, t\rangle \to \hat{A}_z\langle x, z\rangle\big)$,
- $\forall x \big(x \text{ codes an ordinal} \to \exists y \big(\hat{A}_z\langle x, y\rangle \vee \hat{A}_z\langle y, x\rangle\big)\big)$,
- $\forall x \, \forall y \big(|x| = |y| \to \neg \hat{A}_z\langle x, y\rangle\big)$, and
- $\forall x \, \exists n \big(\neg \hat{A}_z\langle x_{n+1}, x_n\rangle\big)$, where we identify reals with ω-sequences of reals in some recursive way.

If this holds, the transitive collapse $\pi \colon \omega_1 \to \gamma$ of (ω_1, A_z) is the claimed bijection, we denote the ordinal γ by γ^z and also refer to this $\Pi^1_4(z)$-statement as "γ^z exists."

Given z and y such that γ^z and γ^y exist, the statement $\gamma^z \leq \gamma^y$ is $\Delta^1_5(z, y)$: We can state it in a $\Sigma^1_5(z, y)$-way since $\gamma^z \leq \gamma^y$ iff there is an order preserving injection $\pi \colon (\omega_1, A_z) \to (\omega_1, A_y)$. We code $\pi \colon \omega_1 \to \omega_1$ by A_t, where

$$\forall a \, \forall b \, \forall c \, \forall d \big(|a| = |b| \wedge |c| \neq |d| \wedge \hat{A}_t\langle a, c\rangle \to \neg \hat{A}_t\langle b, d\rangle\big) \wedge \forall a \, \exists b \big(\hat{A}_t\langle a, b\rangle\big).$$

Given such map π, it is order preserving as claimed iff

$$\forall x \, \forall y \, \forall u \, \forall v \big(\hat{A}_t\langle x, u\rangle \wedge \hat{A}_t\langle y, v\rangle \wedge \hat{A}_z\langle x, y\rangle \to \hat{A}_y\langle u, v\rangle\big).$$

To claim that there is a t such that A_t is as above is $\Sigma^1_5(z, y)$. Similarly, $\gamma^z \leq \gamma^y$ iff every order preserving injection $\pi\colon (\omega_1, A_y) \to (\omega_1, A_z)$ is onto (which is to say, if $\gamma^y \leq \gamma^z$ then $\gamma^y = \gamma^z$), and this gives a $\Pi^1_5(z, y)$-formulation of the statement.

Let r and s code ordinals, say β and α, respectively, and let A_z code a bijection $\pi\colon \omega_1 \to \gamma^z$. Then we can express that $\beta = \operatorname{ot}\pi``\alpha$ in a $\Sigma^1_4(r, s, z)$-way: We must express that there is an order preserving bijection between (ω, r) and $\big(\alpha, A_z \cap (\alpha \times \alpha)\big)$. For this, identify ω_1 with $\omega \times \omega_1$, and $\mathbb{R}$ with $\omega \times \mathbb{R}$. We can then state that there is such a bijection by claiming that there is a real t such that the conjunction of the following clauses holds:

- $\forall n\, \forall x \big(\hat{A}_t\langle n, x\rangle \to x \text{ codes an ordinal} \wedge |x| < \alpha\big)$,
- $\forall n\, \exists x \big(\hat{A}_t\langle n, x\rangle\big) \wedge \forall x \big(|x| < \alpha \to \exists n(\hat{A}_t\langle n, x\rangle)\big)$,
- $\forall n\, \forall m\, \forall x\, \forall y \big[\big(\hat{A}_t\langle n, x\rangle \wedge \hat{A}_t\langle n, y\rangle \to |x| = |y|\big) \wedge \big(\hat{A}_t\langle n, x\rangle \wedge \hat{A}_t\langle m, x\rangle \to n = m\big) \wedge \big(|x| = |y| \wedge \hat{A}_t\langle n, x\rangle \to \hat{A}_t\langle n, y\rangle\big)\big]$, and
- $\forall n\, \forall m\, \forall x\, \forall y \big[\hat{A}_t\langle n, x\rangle \wedge \hat{A}_t\langle m, y\rangle \to \big(r\langle n, m\rangle \leftrightarrow \hat{A}_z\langle x, y\rangle\big)\big]$.

Recall now that $(S_n\colon n < \omega)$ is Δ^1_3-in-the-codes. It follows that we can express "t codes the ordinal γ_x as witnessed by the club A_a" in a $\Pi^1_5(x, t, a)$-way by saying that

- A_t codes a bijection $\pi\colon \omega_1 \to \gamma^t$,
- A_a is club,
- $S_x \cap A_a = \{\alpha \in A_a\colon \operatorname{ot}(\pi``\alpha) \in S_0\}$ (this is a $\Pi^1_4(x, a, t)$-statement, see below), and
- for all reals q and b such that A_q codes a bijection $\pi_1\colon \omega_1 \to \gamma^q$ and A_b is a club and $S_x \cap A_b = \{\alpha \in A_b\colon \operatorname{ot}(\pi_1``\alpha) \in S_0\}$, it is then the case that $\gamma^t \leq \gamma^q$.

If this holds, then $\gamma_x = \gamma^t$. To see that

$$S_x \cap A_a = \{\alpha \in A_a\colon \operatorname{ot}(\pi``\alpha) \in S_0\}$$

is as claimed, notice that it can be expressed as follows: For all reals y, if $\hat{A}_a(y)$ holds, then

$$\big[\hat{S}_x(y) \wedge \forall u\big(|u| = \operatorname{ot}\pi``|y| \to \hat{S}_0(u)\big)\big] \vee \big[\neg\hat{S}_x(y) \wedge \forall u\big(|u| = \operatorname{ot}\pi``|y| \to \neg\hat{S}_0(u)\big)\big],$$

where $\hat{S}_x = \bigcup\{\hat{S}_{n+1}\colon n \in x\}$ and we denote by $\hat{S}_n$ the set of codes for ordinals in S_n.

We can thus define the well-ordering by saying that $x < y$ iff $x \neq y$ and there are reals t, u, a, b such that γ^t and γ^u exist, $\gamma^t = \gamma_x$ as witnessed by the club A_a, $\gamma^u = \gamma_y$ as witnessed by the club A_b, and $\gamma^t \leq \gamma^u$. This is a Σ^1_6-statement about x and y, and the proof is complete.

REMARK 3.12. It is apparent from the proof that it suffices that $L[\mathscr{E}]$ has a cardinal λ which is both Σ_1-reflecting and 3-strong. Recall from [13, Definition 2.2] that a cardinal κ is Σ_1-*reflecting* iff κ is regular and for all $a \in H_\kappa$, all formulas $\varphi(x)$ and all $\xi \geq \kappa$, if $H_\xi \models \varphi(a)$ then there is $\delta < \lambda$ such that $a \in H_\delta$ and $H_\delta \models \varphi(a)$.

In effect, the argument we have presented for $\mathsf{SPFA}(\mathfrak{c})$ only requires λ to be 3-strong while the argument for BSPFA^{++} only requires that it be Σ_1-reflecting. Since λ is still a limit of measurable cardinals, ψ_{AC} holds in $L[\mathscr{E}]^{\mathbb{P}}$. The definition and complexity of the well-ordering, and both $\underset{\sim}{\mathbf{\Delta}}^1_3(\mathscr{L})$ and $\underset{\sim}{\mathbf{\Delta}}^1_3(\mathscr{M})$ require that Σ^1_3-absoluteness holds between the final model $L[\mathscr{E}]^{\mathbb{P}}$ and all the intermediate models

$L[\mathscr{E}]^{\mathbb{P}_\alpha}$, $\alpha < \lambda$. To see that this is the case, notice that Σ^1_3-absoluteness between $L[\mathscr{E}]^{\mathbb{P}_\alpha}$ and any $L[\mathscr{E}]^{\mathbb{P}_\beta}$, $\alpha < \beta < \lambda$, holds by Theorem 1.8 since all sets in $L[\mathscr{E}] \cap V_\lambda$ have sharps. The forcing $\mathbb{P}$ has the property that any real it adds is added by an initial stage $\mathbb{P}_\beta$, and we may without loss of generality assume that $\beta > \alpha$. Clearly, any Σ^1_3-statement in $L[\mathscr{E}]^{\mathbb{P}}$ (with parameters in $L[\mathscr{E}]^{\mathbb{P}_\alpha}$) is witnessed by one such real, and Σ^1_3-absoluteness between $L[\mathscr{E}]^{\mathbb{P}_\alpha}$ and $L[\mathscr{E}]^{\mathbb{P}_\beta}$ guarantees that the Σ^1_3-statement holds in $L[\mathscr{E}]^{\mathbb{P}_\alpha}$ as well.

§4. A question. We do not know if something like the above can even produce a $\underset{\sim}{\mathbf{\Sigma}}^1_5$-well-ordering, but it seems difficult to be able to turn this definition into one of complexity $\underset{\sim}{\mathbf{\Sigma}}^1_4$ if, as in our case, the model is obtained by set forcing, given that Σ^1_3-absoluteness holds: It is conceivable that for $(S_n : n < \omega)$ as above and for some real x, there is $\hat{\gamma}_x < \gamma_x$ such that for some stationary costationary set T,

$$[S_x \cup T]_{\mathrm{NS}_{\omega_1}} = [\![\hat{\gamma}_x \in j(S_0)]\!]_{\mathrm{RO}(\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1})}.$$

If such is the case, it seems like the value of γ_x can be lowered at least to $\hat{\gamma}_x$ by shooting a club that misses T while preserving the sets S_n. But Σ^1_3-absoluteness seems to prevent this from happening, if the well-ordering is $\underset{\sim}{\mathbf{\Sigma}}^1_4$.

This does not mean that a $\underset{\sim}{\mathbf{\Sigma}}^1_4$- or a $\underset{\sim}{\mathbf{\Sigma}}^1_5$-well-ordering in the statement of Theorem 1.6 is impossible, but if one of complexity $\underset{\sim}{\mathbf{\Sigma}}^1_4$ can be produced perhaps class forcing techniques are required, the problem becoming that of adding solutions to a projective (Π^1_3) predicate via projective (Π^1_3) singletons.

QUESTION 4.1. *Is the existence of $\underset{\sim}{\mathbf{\Sigma}}^1_4$-well-orderings of $\mathbb{R}$ consistent with ψ_{AC} + $\mathsf{SPFA}(\mathfrak{c})+\mathsf{BSPFA}^{++}+\underset{\sim}{\mathbf{\Delta}}^1_3(\mathscr{L})+\underset{\sim}{\mathbf{\Delta}}^1_3(\mathscr{M})$? Does this depend on whether Σ^1_3-absoluteness holds?*

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