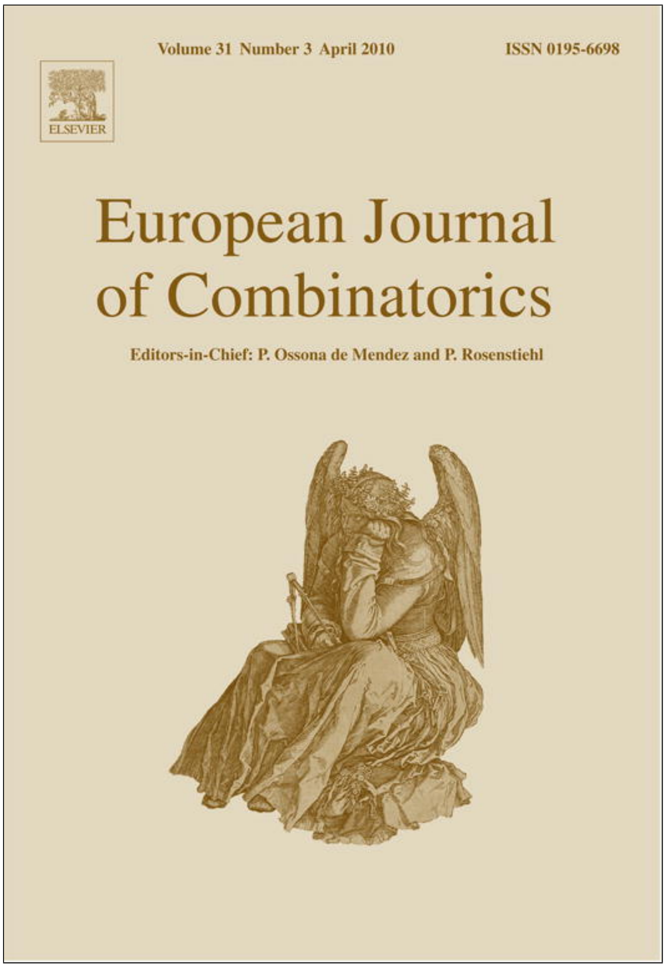
Volume 31 Number 3 April 2010
ISSN 0195-6698
ELSEVIER
European Journal of Combinatorics
Editors-in-Chief: P. Ossona de Mendez and P. Rosenstiehl

European Journal of Combinatorics 31 (2010) 803–812

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Regressive functions on pairs

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A R T I C L E I N F O

Article history:
Received 24 October 2007
Accepted 23 July 2009
Available online 18 August 2009

A B S T R A C T

We compute an explicit upper bound for the regressive Ramsey numbers by a combinatorial argument, the corresponding function being of Ackermannian growth. For this, we look at the more general problem of bounding $g(n, m)$, the least l such that any regressive function $f : [m, l]^{[2]} \to \mathbb{N}$ admits a min-homogeneous set of size n. An analysis of this function also leads to the simplest known proof that the regressive Ramsey numbers have a rate of growth at least Ackermannian. Together, these results give a purely combinatorial proof that, for each m, $g(\cdot, m)$ has a rate of growth precisely Ackermannian, considerably improve the previously known bounds on the size of regressive Ramsey numbers, and provide the right rate of growth of the levels of g. For small numbers we also find bounds on their values under g improving those provided by our general argument.

1. Introduction

Throughout this paper, $\mathbb{N} = \{0, 1, \ldots\}$. For $1 \leq n, k \leq m$, let $m \to (n)^k_{reg}$ be the following assertion:

Whenever $f : [1, m]^{[k]} \to [0, m-k]$ is regressive, there is $H \in [1, m]^{[n]}$ min-homogeneous for f.

Similarly, for $X \subseteq \mathbb{N}$ infinite, let $X \to (\mathbb{N})^k_{reg}$ mean that for every regressive $f : X^{[k]} \to \mathbb{N}$ there is $H \subseteq X$ infinite and min-homogeneous for f. Here,

- $X^{[k]}$ is the collection of k-sized subsets of X.

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[1] This paper was prepared while the author was the Harry Bateman Research Instructor at the California Institute of Technology.

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2009.07.010

- $f : X^{[k]} \to \mathbb{N}$ is regressive iff $f(s) < \min(s)$ whenever $s \in X^{[k]}$ and $\min(s) > 0$ (where $\min(s)$ is the least element of s).
- For such an f, $H \subseteq X$ is min-homogeneous for f iff $0 \notin H$ and, whenever $s, t \in H^{[k]}$ and $\min(s) = \min(t)$, then $f(s) = f(t)$.
- $[n, m] = \{n, n+1, \ldots, m\}$. Similarly for other interval notation.

The following is the main result of Kanamori–McAloon [5]:

Theorem 1.1. 1. *For any $k, n \in \mathbb{N}$, there is m such that $m \to (n)^k_{reg}$.*
2. *Item 1 is not a theorem of Peano Arithmetic* PA.

In fact, in Kanamori–McAloon [5] a level-by-level correspondence is established between the values of k and the amount of induction required to prove the existence of the function that to n assigns the least m as in Theorem 1.1.1; see Carlucci–Lee–Weiermann [2] for more on this.

In this paper, I only deal with $k = 2$ although, in Section 3, I present a short proof of Theorem 1.1.1. In Section 4, I show that

$$g(n) = \text{ least } l \text{ such that } l \to (n)^2_{reg}$$

is provably total in PA. In fact, I provide an explicit (recursive) upper bound for $g(n)$, thus showing by purely elementary means that its rate of growth is at most Ackermannian.

To state the result, let $g(n, m)$ be the least l such that for any regressive

$$f : [m, l]^{[2]} \to [0, l-2],$$

there is a min-homogeneous set for f of size n. (From now on, all mentions of g refer to this two-variable function.) Clearly $g(n, m) \le g(n, m+1)$, $g(2, m) = m + 1$ and, by the pigeonhole principle, $g(3, m) = 2m + 1$.

Let $G(n, m)$ be the least l such that for any regressive $f : [m, l]^{[2]} \to [0, l-2]$, there is a min-homogeneous set for f of size n whose minimum element is m. It may not be immediate that G is well defined, but this is addressed by Remark 3.3 and the proof of Theorem 4.1.

We have $G(2, m) = g(2, m)$, $G(3, m) = g(3, m)$, $G(n+1, 1) = g(n+1, 1) = g(n, 2)$ and, in general, $g(n, m) \le G(n, m)$. Finally, set $g^0(n, m) = m$ and $g^{k+1}(n, m) = g(n, g^k(n, m))$. We then have:

Theorem 1.2. 1. $G(4, m) = 2^m(m + 2) - 1$.
2. *Let $\alpha_{-1} = 0$ and, for $0 \le i < m$, let $d_i = g^i(4, m+1)$ and*

$$\alpha_i = (\alpha_{i-1} + m + 3 + i)(2^{d_i} - 1).$$

Then $g(5, m) \le (2m + 1) + \sum_{i=0}^{m-1} \alpha_i$.
3. *For all n, there is a constant c_n such that $G(n, m) < A_{n-1}(c_n m)$ for almost all m.*

Here, $A_n = A(n, \cdot)$ where A is Ackermann's function, see Section 2. Theorem 1.2.2 is proven by adapting the argument of Blanchard [1, Lemma 3.1] (that bounds $g(5, 2)$) to the more general problem of bounding $g(5, m)$. In Kojman–Shelah [7], explicit lower bounds for g are computed, showing that g is at least of Ackermannian growth (our notion of "Ackermannian growth" is more restrictive than that of Kojman–Shelah [7] or Kojman–Lee–Omri–Weiermann [6], and is discussed in Section 2). In Section 5, I find lower bounds for $G(n, m)$ and $g(n, m)$ in terms of iterates of $g(n-1, \cdot)$, and conclude:

Theorem 1.3. $g(n, m) \ge A_{n-1}(m - 1)$ *for all* $n \ge 2$.

The proof of Theorem 1.3 is simpler and shorter than the proofs of lower bounds in Kojman–Shelah [7] and Kojman et al. [6], and increases these bounds significantly. Thus the results of Sections 4 and 5 combine to give a very accessible and purely combinatorial proof of the result obtained in Kanamori–McAloon [5] by model theoretic methods, that g is not provably total in Primitive Recursive Arithmetic PRA, but is "just shy" of it; in fact, the argument gives that, for each m, the function $g(\cdot, m)$ has an Ackermannian rate of growth. These results also establish the rate of growth of the function $g(n, \cdot)$ as being precisely that of the $(n-1)$st level of the Ackermann hierarchy of fast growing functions.

In the literature, the values of g (more precisely, the values of $g(\cdot, 2)$) are referred to as "regressive Ramsey numbers." In Section 6, I improve the upper bound for $g(4, m)$ and show:

Theorem 1.4. $g(4, 3) = 37$.

I also improve the upper bound for $g(4, 4)$ provided by the general argument of Section 6. The figures so obtained improve the previously known bounds for small regressive Ramsey numbers obtained in Blanchard [1] and Kojman et al. [6].

I occasionally abuse notation by writing $f(t_1, t_2)$ for $f(t)$ where $t_1 < t_2$ and $t = \{t_1, t_2\}$.

2. Preliminaries on Ackermannian functions

In this section I collect several standard results about Ackermannian growth; notice that the notion I use is more restrictive than the version used in Kojman–Shelah [7] or Kojman et al. [6], where a function is called Ackermannian simply if it eventually dominates each primitive recursive function.

Definition 2.1. Given functions $g, h : \mathbb{N} \to \mathbb{N}$, say that h *eventually dominates* g, in symbols $g <_* h$, iff $g(m) < h(m)$ for all but finitely many values of m.

Definition 2.2. Ackermann's function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by double recursion as follows:

- $A(0, m) = m + 1$.
- $A(n, 0) = A(n - 1, 1)$ for $n > 0$.
- $A(n, m) = A(n - 1, A(n, m - 1))$ for $n, m > 0$.

Let $\text{Ack}(n) = A(n, n)$ and $A_n = A(n, \cdot)$. Sometimes, in the literature, it is Ack that is referred to as Ackermann's function. This is the standard example of a recursive but not primitive recursive function. The version presented above is due to Rafael Robinson and Rózsa Péter, see Robinson [8]. Notice that $A_1(m) = m + 2$, $A_2(m) = 2m + 3$, A_3 has an exponential rate of growth and A_4 grows like a tower of exponentials.

Definition 2.3. Let $f_0(m) = m + 1$ and $f_{n+1}(m) = f_n^m(m)$ where the superindex indicates that f_n is iterated m times. Continue this hierarchy by letting $f_\omega(m) = f_m(m)$ and $f_{\omega+1}(m) = f_\omega^m(m)$.

Notice that what in Kojman et al. [6] is called Ackermann's function is the map $A'(n, m) = f_{n-1}(m)$.

Definition 2.4. A function $f : \mathbb{N} \to \mathbb{N}$ is *(precisely) of Ackermannian growth* if and only if there are constants $c, C > 0$ such that for all but finitely many m, $f_\omega(cm) \le f(m) \le f_\omega(Cm)$.

Similarly, say that a function's rate of growth is like that of the nth level of the Ackermann hierarchy if there are constants $c, C > 0$ such that for all but finitely many m, $A_n(cm) \le f(m) \le A_n(Cm)$.

(Compare with Graham–Rothschild–Spencer [4, Section 2.7], where the relevant notion is called *Ackermannic*.)

The following two lemmas are standard and collect together several folklore results; see for example Graham–Rothschild–Spencer [4] and Cori–Lascar [3].

Lemma 2.5. 1. *For all n, $A_n < A_{n+1}$ and $f_n <_* f_{n+1}$. In fact, for any $C > 0$ and almost all m, $A_n(Cm) < A_{n+1}(m)$ for $n > 0$, and $f_n(Cm) < f_{n+1}(m)$ for all n.*
2. *For all $n > 0$, $A_{n+1} <_* f_n$ and $f_n(m) < A_{n+1}(cm)$ for some constant $c = c_n$ and all m.*
3. *f_ω and Ack are of Ackermannian growth.* □

More precise quantitative versions of the above are possible, but Lemma 2.5 as stated suffices for our purposes.

Lemma 2.6. 1. *If f is of Ackermannian growth, it eventually dominates each primitive recursive function. In particular, it eventually dominates each f_n.*
2. *If f is of Ackermannian growth then it is eventually dominated by $f_{\omega+1}$.*
3. *There is a function f that eventually dominates each f_n and is eventually dominated by $f_{\omega+1}$ but is not of Ackermannian growth.*
4. *If g, h are strictly increasing primitive recursive functions and f is of Ackermannian growth, then so is $g \circ f \circ h$.* □

3. Regressive functions

I start by proving the infinite version of Theorem 1.1.1. This is also done in Kanamori–McAloon [5], but the argument to follow is easier (in Kanamori–McAloon [5] this is accomplished using the Erdős–Rado canonization theorem). The proof of Theorem 1.2 in Section 4 was obtained by trying to produce a finitary and effective version of this argument for $k = 2$.

Lemma 3.1. *If $X \subseteq \mathbb{N}$ is infinite, then for any k, $X \to (\mathbb{N})^k_{reg}$.*

Proof. Let $f : X^{[k]} \to \mathbb{N}$ be regressive. Without loss, $k > 1$. Define a decreasing sequence of infinite subsets of X, $X \setminus \{0\} = H_0 \supset H_1 \supset H_2 \supset \cdots$ such that, letting $m_n = \min H_n$, then $(m_n)_{n\geq 0}$ is strictly increasing, as follows: Given H_n, let

$$\varphi : (H_n \setminus \{m_n\})^{[k-1]} \to [0, m_n - 1]$$

be the function $\varphi(s) = f(\{m_n\} \cup s)$. By Ramsey's theorem, there is H_{n+1} infinite and homogeneous for φ.

Then $\{ m_n : n \in \mathbb{N} \}$ is min-homogeneous for f. □

Theorem 1.1.1 follows now from a standard compactness argument:

Corollary 3.2. $\forall n \forall k \exists l\, (l \to (n)^k_{reg})$.

Proof. Fix n and k counterexamples to the corollary. For each $m \geq n, k$, it follows that there are regressive functions $f : [1, m]^{[k]} \to [0, m - k]$ without min-homogeneous sets of size n. Consider the collection $\mathcal{T}$ of all these functions, ordered by extension: Given $f_1, f_2 \in \mathcal{T}$, $f_1 : [1, m_1]^{[k]} \to [0, m_1 - k]$, $f_2 : [1, m_2]^{[k]} \to [0, m_2 - k]$, set $f_1 < f_2$ iff $m_1 < m_2$, and $f_2 \upharpoonright [1, m_1]^{[k]} = f_1$. Then $(\mathcal{T}, <)$ is an infinite finitely branching tree so, by König's lemma, it has an infinite branch. The functions along this branch fit together into a regressive function $f : \mathbb{N}^{[k]} \to \mathbb{N}$ which contradicts Lemma 3.1 since it does not even admit min-homogeneous sets of size n. □

Remark 3.3. Notice that using this argument one can easily show that $G(n, m)$ is well defined. Our argument next section will also show this.

4. An Ackermannian upper bound for *G*

Here I prove Theorem 1.2.3; the argument resembles the "color focusing" technique from Ramsey theory.

Theorem 4.1. *For each fixed m, $G(n, m)$ is bounded by a function of Ackermannian growth. In particular, so is $g(n, 2) \leq G(n, 2)$.*

Proof. I find an upper bound for the function $G(n, \cdot)$ by induction on n. In order to do this, I introduce numbers $s_i = s(i, n, m)$ for all $n \geq 4$, $m \geq 2$, and $1 \leq i \leq m$, and argue that $G(n, m) \leq s(m, n, m)$.

Fix $n \geq 4$. The numbers s_i are computed in terms of the function $G(n - 1, \cdot)$. Fix m, which we may assume is at least 2.

Define $s(1, n, m), \ldots, s(m, n, m)$ and $t_0, t_1, \ldots, t_{m-1}$ recursively as follows.

- Let $t_0 = m + 1$.
- Let $s_1 = g(n - 1, t_0)$ and, for $1 \leq i < m$, let $s_{i+1} = G(n - 1, t_i)$.
- For $1 \leq j \leq m$, let $B^{n,m}_j = B_j = \bigcup_{i=1}^{j} [t_{i-1}, s_i]$, and denote by $\prod B_j$ the Cartesian product $\prod_{i \in B_j} [0, i - 1]$.
- For $1 \leq j < m$, let $t_j = (j + 1) \times |\prod B_j|$.

We claim that $G(n, m) \leq s(m, n, m)$. To see this, suppose a regressive function $f : [m, s_m]^{[2]} \to [0, s_m - 2]$ is given.

Fix j, $1 < j \leq m$. Suppose $f(m, \cdot) \upharpoonright B_j$ takes at most j values. (This holds trivially for $j = m$.) We claim that either there is a min-homogeneous set for f of size n contained in $\{m\} \cup B_j$ whose minimum element is m, or else $f(m, \cdot) \upharpoonright B_{j-1}$ takes at most $j - 1$ values.

Consider the regressive function

$$\psi : [t_{j-1}, s_j]^{[2]} \to [0, s_j - 2]$$

given by

$$\psi(u) = \begin{cases} f(u) & \text{if } u_1 > t_{j-1}, \\ \langle f(l, u_2) : l \in \{m\} \cup B_{j-1} \rangle & \text{if } u_1 = t_{j-1}, \end{cases}$$

where $\langle \cdots \rangle$ is a bijection from the Cartesian product $C_j \times \prod B_{j-1}$ onto $[0, t_{j-1})$, where $C_j \subset [0, m-1]$ has size j and contains the possible values that $f(m, \cdot) \restriction B_j$ can take.

Then (by definition of s_j) there is a set $\{a_1, \ldots, a_{n-2}\} \subseteq [t_{j-1}+1, s_j]$ that is min-homogeneous for f and such that for all $k \in \{m\} \cup B_{j-1}$, $\{k, a_1, \ldots, a_{n-2}\}$ is also min-homogeneous for f. Let $f(m, a_1) = c$. If $f(m, k) = c$ for any $k \in B_{j-1}$, then $\{m, k, a_1, \ldots, a_{n-2}\}$ is the min-homogeneous set we are looking for. Otherwise, $f(m, \cdot) \restriction B_{j-1}$ takes at most $j-1$ values, as claimed.

There is therefore no loss in assuming that $f(m, \cdot) \restriction B_1$ is constant. But then, by definition of s_1, there is $\{a_1, \ldots, a_{n-1}\} \subseteq B_1$ min-homogeneous for f. Then $\{m\} \cup \{a_1, \ldots, a_{n-1}\}$ is also min-homogeneous, and we are done.

Define a function $H(n, m)$ as follows: $H(n, \cdot) = G(n, \cdot)$ for $n \leq 4$ (see also Fact 5.3); in the argument above, let s_i' be the function resulting from replacing $G(n-1, \cdot)$ with $H(n-1, \cdot)$ in the definition of s_i, and let $H(n, m) = s'(m, n, m)$, so clearly $G \leq H$. It is easy to see, using standard arguments (or consider the proof of Theorem 1.2.3 below) that $n \mapsto H(n, m)$ (for any fixed m) is of Ackermannian growth. This completes the proof. □

Remark 4.2. Since the argument above only requires f to be defined on

$$(\{m\} \cup B_m^{n,m})^{[2]},$$

it follows (by "translation") that $g(n, m) \leq m + |B_m^{n,m}|$.

That $G(4, m) = 2^m(m+2) - 1$ is shown in Fact 5.3, and the upper bound on $g(5, \cdot)$ is shown in Theorem 7.1. Using this (all I need is that $G(4, m)$ has an exponential rate of growth) and the argument of Theorem 4.1, Theorem 1.2.3 follows easily:

Proof. Use the notation of the proof above, and argue by induction on $n \geq 5$ since the result is clear for $n \leq 4$ from the explicit formulas for $G(n, \cdot)$. Notice the easy estimate $l! < 2^{l(l-1)/2}$ and the obvious inequality $s(i+1, n, m) = s_{i+1} \leq G(n-1, s_i!)$ for $i < m$. From this and Fact 5.3 we have that for $n = 5$ there is a constant c_5 such that s_i is bounded by a tower of two's of length $c_5 i$ applied at m,

$$s_i \leq 2^{2^{\cdot^{\cdot^{2^m}}}}.$$

In fact any c_5 slightly larger than 3 suffices (with room to spare). This proves the result for $n = 5$; for $n > 5$ use Lemma 2.5 and proceed by a straightforward induction to show that $c_{n-1} = n - 1$ suffices (and therefore for each m, $g(\cdot, m)$ has a rate of growth precisely Ackermannian). □

Question 4.3. *Can the value of the constants c_n be significantly improved? This seems to require a more careful analysis than the one above, perhaps combined with fine detail considerations, as in the proof of Theorem 7.1.*

5. Lower bounds for g and G

Here I prove Theorem 1.3.

Theorem 5.1. 1. $G(n+1, m) \geq g^m(n, m+1)$.
2. $g(n+1, m+1) \geq g(n, g(n+1, m)+1)$. *In particular, for $n \geq 2$ and $m \geq 1$, $g(n, m) \geq A_{n-1}(m-1)$, the inequality being strict for $n > 2$ and, for example, $g(4, m) > 2^{m+2}$ for $m > 1$.*

Proof. I exhibit a regressive function $f : [m, g^m(n, m+1) - 1]^{[2]} \to \mathbb{N}$ without min-homogeneous sets of size $n+1$ whose minimum element is m. Start by choosing regressive functions

$$F_k : [g^k(n, m+1), g^{k+1}(n, m+1) - 1]^{[2]} \to \mathbb{N}$$

without min-homogeneous sets of size n, for $k < m$; this is possible by definition of $g(n, \cdot)$. Now set, for $m < a \le g^m(n, m+1) - 1$,

$$f(m, a) = k \iff g^k(n, m+1) \le a < g^{k+1}(n, m+1),$$

and, for such a, and $b \in (a, g^{k+1}(n, m+1) - 1]$,

$$f(a, b) = F_k(a, b).$$

Define $f(a, b)$ for other values of a and b arbitrarily (below a). This function works, for if $\min(H) > m$ and $\{m\} \cup H$ is min-homogeneous for f, then H is completely contained in some interval

$$[g^k(n, m+1), g^{k+1}(n, m+1))$$

for some $k < m$, but then H is min-homogeneous for F_k, so $|H| < n$.

I now prove item 2. Let $F_m : [m, g(n+1, m)]^{[2]} \to \mathbb{N}$ be a regressive function without min-homogeneous sets of size $n+1$, and let

$$h_m : [g(n+1, m) + 1, g(n, g(n+1, m) + 1)]^{[2]} \to \mathbb{N}$$

be a regressive function without min-homogeneous sets of size n. Define

$$F_{m+1} : [m+1, g(n, g(n+1, m) + 1)]^{[2]} \to \mathbb{N}$$

by

$$F_{m+1}(a, b) = \begin{cases} F_m(a-1, b-1) & \text{if } b \le g(n+1, m), \\ a - 1 & \text{if } a \le g(n+1, m) < b, \\ h_m(a, b) & \text{if } g(n+1, m) < a. \end{cases}$$

Then F_{m+1} is regressive. If H is min-homogeneous for F_{m+1} and $|H| \ge 2$, let $a = \min(H)$ and $b = \min(H \setminus \{a\})$. If $b \le g(n+1, m)$ then $F_{m+1}(a, b) = F_m(a-1, b-1) < a-1$ so $H \subseteq [m+1, g(n+1, m)]$ and $\{h - 1 : h \in H\}$ is min-homogeneous for F_m, so $|H| \le n$.

If $g(n+1, m) < b$ then $H \setminus \{a\}$ is min-homogeneous for h_m, so $|H \setminus \{a\}| < n$ and $|H| < n+1$ in this case as well. □

Remark 5.2. Notice that for $n = 3$, the argument of Theorem 5.1.1 describes (up to trivial renamings) all the examples of regressive functions $f : [m, g^m(3, m+1) - 1]^{[2]} \to \mathbb{N}$ not admitting min-homogeneous sets of size 4 with minimum element m. It is easy now to give an example of a regressive $f : [2, 14]^{[2]} \to \mathbb{N}$ witnessing $14 \not\to (5)^2_{reg}$:

$$f(i, j) = \begin{cases} j - i - 1 \pmod{i} & \text{if } i \ge 6, \\ 0 & \text{if } \begin{array}{l} i = 2 \text{ and } j \le 6, \\ i \in [3, 5] \text{ and } j = i + 1, \end{array} \\ 1 & \text{if } \begin{array}{l} i = 2 \text{ and } 7 \le j, \\ i = 3 \text{ and } j \in \{5, 7, 8\}, \\ i \in \{4, 5\} \text{ and } j = i + 1, \end{array} \\ 2 & \text{if } \begin{array}{l} i = 3 \text{ and } j \in \{6\} \cup [9, 14], \\ i = 4 \text{ and } j = 7, \\ i = 5 \text{ and } 8 \le j, \end{array} \\ 3 & \text{if } i = 4 \text{ and } 8 \le j. \end{cases}$$

I leave to the reader the easy verification that this example works; in Theorem 6.1.2, I analyze a more difficult example witnessing $g(4, 3) \ge 37$. See Blanchard [1] for an analysis of a different example also witnessing $g(4, 2) \ge 15$; the function I have presented is closer in spirit to the other constructions in this paper.

Now I prove Theorem 1.2.1:

Fact 5.3. $G(4, m) = 2^m(m+2) - 1$.

Proof. Notice that $2^m(m+2) - 1 = g^m(3, m+1) \le G(4, m)$ by Theorem 5.1.1. Suppose $f : [m, 2^m(m+2) - 1]^{[2]} \to \mathbb{N}$ is regressive. A straightforward induction on $k \le m$ shows that either $f(m, \cdot) \restriction [m+1, 2^k(m+1) + 2^k - 1]$ takes at least $k+1$ values, or else f admits a min-homogeneous set $A \in [m, 2^k(m+1) + 2^k - 1]^{[4]}$ with $m \in A$ (see also the proof of Theorem 6.1.1 for a more detailed presentation of a similar approach). When $k = m$, this shows that $G(4, m) \le 2^m(m+2) - 1$. □

Remark 5.4. Thus, $g(4, 2) = G(4, 2) = 15$. In the next section, I improve the upper bound for $g(4, m)$, $m > 2$.

Corollary 5.5. $g(5, 2) > 2^{18}$.

This significantly improves the bound $g(5, 2) \ge 195$ claimed in Blanchard [1].

Proof. $g(5, 2) \ge g(4, g(5, 1) + 1) = g(4, 16) > 2^{18}$. □

Remark 5.6. In fact, by Theorem 6.1.2, $g(4, 3) = 37$, so $g(4, m) \ge 5 \times 2^m - 3$ for $m \ge 3$, and $g(5, 2) \ge 5 \times 2^{16} - 3$.

Theorem 5.1.2 also improves significantly the bound $g(81, 2) > f_{51}(2^{2^{274}})$ obtained in Kojman et al. [6, Claim 2.32] (here, f_{51} is as in Section 2; to see that the new bound is an improvement, a slightly more precise version of Lemma 2.5 is necessary).

6. Bounds for $g(4, \cdot)$

From Section 5 it follows that $g(4, m) \le 2^m(m+2) - 1$. Here I improve this bound and prove Theorem 1.4.

Theorem 6.1. 1. *For $m \ge 2$, $g(4, m) \le 2^m(m+2) - 2^{m-1} + 1$.*
2. $g(4, 3) = 37$.
3. $g(4, 4) \le 85$.

Proof. I have already shown that $g(4, 2) = 15$. Assume $m \ge 3$, let

$$n = 2^m(m+2) - 2^{m-1} + 1,$$

and suppose a regressive $f : [m, n]^{[2]} \to \mathbb{N}$ is given. I need to argue that there is $H \in [m, n]^{[4]}$ min-homogeneous for f. For $i < m$, let $a_i = \min\{j : f(m, j) = i\}$ and $C_i = \{j > a_i : f(m, j) = i\}$. One may assume that, as long as the a_i are defined, they occur in order, so $m + 1 = a_0 < a_1 < \cdots$

If $f(m+1, a) = f(m+1, b)$ for $a \ne b$ in C_0, then $H = \{m, m+1, a, b\}$ is as required. Assume now that $f(m+1, \cdot) \restriction C_0$ is injective and, in particular, $|C_0| \le m + 1$.

For $i \in C_0$ let $B_i = \{j > i : f(m+1, j) = f(m+1, i)\}$. I claim that for all $k \in [1, m-2]$, either $a_k \le 2^k(m+2) - 2^{k-1} - 1$, or else there is an H as required and either of the form $\{m, a_i, a, b\}$ for some $i < k$ and some $a, b \in C_i$, or of the form $\{m+1, i, a, b\}$ for some $i \in C_0$ and some $a, b \in B_i$.

The proof is by induction on k. Fix a least counterexample. Then

$$a_t \le 2^t(m+2) - 2^{t-1} - 1$$

for all $t \in [1, k)$ and $1 \le k < m - 1$. Then $a_k \le 2^k(m+2) - 2^{k-1}$. Otherwise, for some $i < k$, $|C_i| > a_i$. If $a_k = 2^k(m+2) - 2^{k-1}$, then $a_t = 2^t(m+2) - 2^{t-1} - 1$ for all $t \in [1, k)$ (or else, again, some C_i for $i < k$ has size larger than a_i). Also, there is some $j \in (2m+1, a_k)$ in C_0. But then $|B_i| > i$ for some $i \in C_0$, and the claim follows: Otherwise,

$$\begin{aligned}\sum_{i \in C_0} |B_i| \le \sum_{i \in [m+2, 2m+1] \cup \{j\}} i &\le \sum_{i=m+2}^{2m+1} i + 2^k(m+2) - 2^{k-1} - 1 \\ &= \frac{3}{2}m(m+1) + 2^k(m+2) - 2^{k-1} - 1 \\ &< n - 2(m+1) = |[2m+2, n] \setminus \{j\}|\end{aligned}$$

because $(3 + 2m)(2^m - 2^k) \ge 3(3 + 2m)2^{m-2} > 3m^2 + 7m$ for $m \ge 3$.

It follows that one may assume $a_{m-1} \leq 2^{m-1}(m+2) - 2^{m-2}$, but then, since $n \geq 2a_{m-1} + 1$, some C_i must have size larger than a_i, and the proof is complete.

Now I show that $g(4, 3) = 37$. The upper bound follows from the argument above. To see that $g(4, 3) \geq 37$, I exhibit a regressive $f : [3, 36]^{[2]} \to \mathbb{N}$ without min-homogeneous sets of size 4. Consider the function f shown below: For $3 \leq i < j \leq 36$, set

$$
f(i,j) = \begin{cases}
j - i - 1 (\mathrm{mod}\ i) & \text{if } \begin{array}{l} i \geq 16, \\ 8 \leq i \leq 15 \text{ and } j \leq 16, \\ 12 \leq i \leq 15 \text{ and } j \leq 19, \\ 4 \leq i \leq 6 \text{ and } j \leq 7, \\ i = 6 \text{ and } j \leq 11, \end{array} \\
0 & \text{if } \begin{array}{l} i = 3 \text{ and } (j \leq 7 \text{ or } j = 17), \\ i = 5 \text{ and } 8 \leq j \leq 11, \\ i = 6 \text{ and } 12 \leq j \leq 16, \\ i = 7 \text{ and } j \leq 12, \end{array} \\
1 & \text{if } \begin{array}{l} i = 3 \text{ and } 8 \leq j \leq 16, \\ i = 4 \text{ and } 8 \leq j \leq 11, \\ i = 5 \text{ and } 12 \leq j \leq 16, \\ i = 6 \text{ and } j = 18, \\ i = 7 \text{ and } j = 13, \end{array} \\
2 & \text{if } \begin{array}{l} i = 3 \text{ and } 18 \leq j, \\ i = 4 \text{ and } j \in [12, 19] \setminus \{17\}, \\ i = 5 \text{ and } j = 17, \\ i = 6 \text{ and } j = 19, \\ i = 7 \text{ and } j = 14, \\ i = 15 \text{ and } 21 \leq j, \end{array} \\
3 & \text{if } \begin{array}{l} i = 4 \text{ and } (j = 17 \text{ or } 20 \leq j), \\ i = 5 \text{ and } 18 \leq j, \\ i = 7 \text{ and } j = 15, \\ i = 11 \text{ and } 17 \leq j \leq 20, \\ i = 14 \text{ and } 20 \leq j, \end{array} \\
4 & \text{if } \begin{array}{l} i = 7 \text{ and } j = 16, \\ i = 10 \text{ and } 17 \leq j \leq 20, \\ i = 11 \text{ and } 21 \leq j, \\ i = 13 \text{ and } 20 \leq j, \\ i = 15 \text{ and } j = 20, \end{array} \\
5 & \text{if } \begin{array}{l} i = 6 \text{ and } (j = 17 \text{ or } 20 \leq j), \\ i = 7 \text{ and } (j = 17 \text{ or } j = 19), \\ i = 9 \text{ and } 17 \leq j \leq 20, \\ i = 10 \text{ and } 21 \leq j, \\ i = 12 \text{ and } 20 \leq j, \end{array} \\
6 & \text{if } \begin{array}{l} i = 7 \text{ and } (j = 18 \text{ or } 20 \leq j), \\ i = 8 \text{ and } 17 \leq j \leq 20, \\ i = 9 \text{ and } 21 \leq j, \end{array} \\
7 & \text{if } i = 8 \text{ and } 21 \leq j.
\end{cases}
$$

To help understand the example somewhat, notice that the argument above shows that one must have $a_1 = 8$ and $a_2 = 18$, $f(i, \cdot)$ must be injective for $i \geq 18$ and similarly $f(i, \cdot) \restriction C_i$ must be injective for $i \in [4, 7]$ and $C_i = \{j > i : f(3, j) = f(3, 4)\}$, or $i \in [8, 16] \cap \{j : f(3, j) = f(3, 8)\}$ and $C_i = [i+1, 17] \cap \{j : f(3, j) = f(3, 8)\}$. If f is any function satisfying these conditions, $a < b < c < d$, and $A = \{a, b, c, d\}$ is min-homogeneous for f, then $a > 3$ and $b < 18$.

The function f displayed above satisfies the conditions just described. Let A as above be a putative min-homogeneous set. Then $a < 16$ since otherwise $f(a, \cdot)$ does not take any value more than twice.

In fact, $a < 12$, since $12 \le a \le 15$ would imply (for the same reason) that $b \ge 18$. If $8 \le a \le 11$, then $b \ge 15$. Since $f(i, \cdot) \restriction D_i$ is injective for $i \in \{15\} \cup [17, 20]$ and $D_i = (i, 20]$, or $i = 16$ and $D_i = [21, 36]$, this is not possible.

If $a = 7$ then $b \notin [8, 12]$ as $f(i, \cdot) \restriction (i, 12]$ is injective for $i \in [8, 12]$. This forces $b \ge 18$.

If $a = 6$ then $b \notin \{7\} \cup [12, 16]$ as $f(b, \cdot) \restriction [\max(b+1, 12), 16]$ is then injective. This forces $b = 17$ but $f(17, \cdot) \restriction [20, 36]$ is injective, so this cannot be the case.

The analysis above already rules out $a = 5$ since $f(6, \cdot) \restriction [8, 11]$ is injective. Since $f(7, \cdot) \restriction [12, 16] \cup \{18, 19\}$ is also injective, it also rules out $a = 4$, completing the argument.

Finally, I argue that $g(4, 4) \le 85$. Let a regressive $f : [4, 85]^{[2]} \to \mathbb{N}$ be given. Use notation as before. Then one can assume (from the argument for item 1) that $a_1 \le 10$. If $a_1 = 10$, since $6 + 7 + 8 + 9 = 30$, one can assume that there is $b \le 40$ such that $f(5, b) = 4$ (while $f(5, j) = j - 6$ for $j \in [6, 9]$). But then there is a min-homogeneous set for f of size 4 with minimum element 5 and maximum at most 81.

If $a_1 \le 9$ then $a_2 \le 21$. If $a_2 = 21$ then one can assume $f(5, j) = j - 6$ for $j \in [6, 8]$ and there are b_1, b_2 with $f(5, b_1) = 3$, $f(5, b_2) = 4$, $b_1 \le 19$ and $b_2 \le 20$. Since $6 + 7 + 8 + 19 + 20 = 60$, there is again a min-homogeneous set of size 4 in this case. If $a_2 \le 20$, then $a_3 \le 42$ and $|C_i| > a_i$ for some $i < 4$. This shows $g(4, 4) \le 85$. $\square$

7. Bounds for $g(5, \cdot)$

In this section I briefly sketch how to adapt the proof of Blanchard [1, Lemma 3.1] to prove the more general statement below, which concludes the proof of Theorem 1.2. The bound for $g(5, 2)$ is smaller than the one in Blanchard [1] because I take advantage of the fact that $g(4, 3) = 37$, as established in Theorem 6.1.2.

Theorem 7.1. *Let m be given. For $i < m$, set $d_i = g^i(4, m + 1)$. Let $\alpha_{-1} = 0$ and $\alpha_i = (\alpha_{i-1} + m + 3 + i)(2^{d_i} - 1)$ for $0 \le i < m$. Then*

$$g(5, m) \le (2m + 1) + \sum_{i=0}^{m-1} \alpha_i.$$

In particular, $g(5, 2) \le 41 \times 2^{37} - 1$.

Proof. Let n be the purported upper bound displayed above and consider a regressive function $f : [m, n]^{[2]} \to \mathbb{N}$. For $i < m$, let

$$B_i = \{x \in [m + 1, n] : f(m, x) = i\}$$

and, if $B_i \ne \emptyset$, set $a_i = \min(B_i)$. Without loss, $a_0 = m + 1 < a_1 < \cdots$. Clearly, we may assume that $a_i \le g^i(4, m + 1) = d_i$ for all those $i < m$ for which a_i is defined. In particular, since n is sufficiently large, we may assume that the a_i are defined for all $i < m$.

Consider $B_{ij} = \{x \in [a_i + 1, n] : f(m, x) = i, f(a_i, x) = j\}$ for $i < m$ and $j < a_i$ and, if $B_{ij} \ne \emptyset$, set $a_{ij} = \min(B_{ij})$. Let $D = \{B_{ij} : B_{ij} \ne \emptyset\}$ and $q = |D|$, so $q \le \sum_{i=0}^{m-1} d_i$. Let $\{C_s : s < q\}$ be the enumeration of D such that, setting $c_s = \min(C_s)$, then the sequence $(c_s : s < q)$ is strictly increasing.

Notice that $a_i \notin C_l$ for any i, l, and $a_i < a_{ij}$ for all i, j such that a_{ij} is defined. For $i < m$, define k_i as the least $k < q$ such that $a_i < c_k$. Then

$$k_i \le \sum_{j=0}^{i-1} a_i \le \sum_{j=0}^{i-1} d_i.$$

I now proceed to find an upper bound l_s on the size of C_s beyond which one is guaranteed to find a min-homogeneous set of size 5. The value of n displayed above is obtained by first observing that

$$[m, n] = \{m\} \cup \{a_i : i < m\} \cup \bigcup_{s=0}^{q-1} C_s,$$

so $n - m + 1 = m + 1 + \sum_{s=0}^{q-1} |C_s|$, and then setting $n \ge 2m + \sum_s l_s + 1$.

To find l_s, notice that

$$[m, c_s] \subseteq \{m\} \cup \{a_i : a_i < c_s\} \cup \bigcup_0^{s-1} C_j \cup \{c_s\},$$

so $c_s - m + 1 \le 2 + (i+1) + \sum_0^{s-1} |C_j|$, where $s \in [k_{i-1}, k_i)$, or

$$c_s \le m + 1 + (i+1) + \sum_0^{s-1} |C_j|.$$

Let $C'_s = C_s \setminus \{c_s\}$. If

$$|C'_s| \ge (m+2) + (i+1) + \sum_0^{s-1} |C_j|,$$

then $f(c_s, \cdot) \restriction C'_s$ is not injective; so there are $d < e$ in C'_s such that $f(c_s, d) = f(c_s, e)$ and $\{m, a_j, c_s, d, e\}$ is min-homogeneous, where $j \le i$ is chosen so that $C_s = B_{jk}$ for some k.

This gives the upper bound $l_s \le (m + i + 3) + \sum_0^{s-1} l_j$; so, by a straightforward induction,

- $l_s \le 2^s(m+3)$ for $s < d_0$,
- $l_s \le 2^{s-d_0}((m+3)(2^{d_0} - 1) + (m+4))$ for $d_0 \le s < d_0 + d_1$,
- and, in general, for $i < m$, and $\sum_{j=0}^{i-1} d_j \le s < \sum_{j=0}^{i} d_j$, we have

$$l_s \le 2^{s-d_{i-1}}((\cdots((m+3)(2^{d_0} - 1) + (m+4))(2^{d_1} - 1) + \cdots)(2^{d_{i-1}} - 1) + (m+3+i)).$$

These upper bounds give the value of n that I started with, and the claimed inequality $g(5, m) \le n$ follows. In the case $m = 2$, it implies

$$\begin{aligned} g(5,2) &\le (2 \times 2 + 1) + (2+3)(2^{2+1} - 1) + (5(2^3 - 1) + 6)(2^{g(4,3)} - 1) \\ &= 40 + 41(2^{37} - 1) = 41 \times 2^{37} - 1. \end{aligned}$$

This completes the proof. □

I conclude with some questions:

Question 7.2. *Is $G(n+1, m) > g^m(n, m+1)$ for $n > 4$?*

Question 7.3. *Is $2^m(m+1) \le g(4, m)$ for all m?*

The proofs of Theorems 6.1 and 7.1 suggest that to fully understand g requires solving the following question:

For any n, m and regressive $f : [m, g(n, m)]^{[2]} \to \mathbb{N}$, set

$$k_f = \min\{\min(H) : H \in [m, g(n,m)]^{[n]} \text{ is min-homogeneous for } f\},$$

and let

$$k(n, m) = \max\{k_f : f : [m, g(n,m)]^{[2]} \to \mathbb{N} \text{ is regressive}\}.$$

Question 7.4. *What is the rate of growth of the function $k(n, m)$?*

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