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Of Puzzles and Partitions: Introducing Partiti

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In each of the five issues for 2017, readers of this MAGAZINE found a PINEMI puzzle. Pinemi is the creation of Vietnamese puzzle enthusiast Thinh Van Duc Lai, who has also designed PARTITI, the puzzle that will run through this year's issues.

Partiti

Partiti is played on a 6×6 grid in which each cell contains a positive integer. To play, place one or more digits into each cell in such a way that the digits in a cell sum to the indicated positive integer and no digit appears more than once in a cell or between cells that are adjacent or share a corner.

The objective of the game can be described as finding unordered integer partitions of the given numbers consisting of distinct parts from $1, 2, \ldots, 9$ (subject to the additional restriction that the partitions for contiguous cells should use different parts). Such an integer partition of n consists of an increasing sequence of positive integers that sum to n. See Figure 1 for an example.

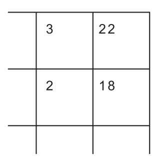

Figure 1 The top-right corner of this month's puzzle. We can solve some of it by noting that the only partition of 2 into distinct parts is 2 itself, and the only such partitions of 3 are $1 + 2$ and 3, but the former is excluded, since we cannot use 2 again. Though more information is needed to see what numbers go into the other two cells, the reader may want to note that $22 + 18 = 1 + 4 + 5 + 6 + 7 + 8 + 9$, the sum of the remaining digits, so all available numbers should be used between these two cells.

The puzzle for this month is at the end of this note. In what follows, we present some basic properties of integer partitions, take a very brief detour through partitions of infinite sets, and conclude with a few words about Partiti's creator Thinh Lai.

Integer partitions

The study of partitions began with Euler. The number of integer partitions of n is often denoted $p(n)$. Hardy and Ramanujan worked out an analytic formula for $p(n)$; the

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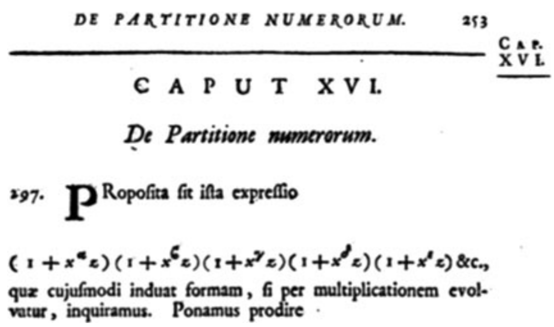
formula takes the form of an infinite series, and even just a few terms produce remarkably accurate approximations. As n increases, $p(n)$ grows faster than a polynomial, but slower than any exponential a^n, $a > 1$. More specifically, $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$, meaning that the quotient of these two expressions approaches one as n approaches infinity. However, puzzlers need not worry, since the possible positive integers in the cells of Partiti are quite modest, the largest potential entry in a cell being $9 + 8 + 7 + 6 + 5 + 4 = 39$, which could only occur in a corner surrounded by 1, 2, and 3.

More relevant than p in this context is the number of partitions of n into distinct parts, usually denoted $q(n)$. For example, $5 = 1 + 4 = 2 + 3$ are all the partitions of 5 into distinct parts, and therefore $q(5) = 3$. By convention, $q(0) = 1$. The function q has a somewhat more modest rate of growth than that of p, namely, $q(n) \sim \frac{1}{4\sqrt[4]{3n^3}} e^{\pi\sqrt{n/3}}$. The sequence $q(0), q(1), \ldots$ is sequence A000009 in the OEIS [3].

The first nontrivial result on this function q is Euler's theorem from 1748 [2] giving us that $q(n)$ coincides with the number of partitions of n into (not necessarily distinct) odd parts. For example, $5 = 1 + 1 + 3 = 1 + 1 + 1 + 1 + 1$ are all such partitions of 5 and, as predicted by the theorem, there are precisely 3 of them. See Figure 2 for the beginning of Euler's work on integer partitions.


DE PARTITIONE NUMERORUM. 253

CAP. XVI.

CAPUT XVI.

De Partitione numerorum.

297. Proposita sit ista expressio

$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z)(1+x^{\varepsilon}z)$ &c., quæ cujusmodi induat formam, si per multiplicationem evolvatur, inquiramus. Ponamus prodire

Figure 2 The opening of chapter 16 of Euler's *Introductio in Analysin Infinitorum* [2]. The book lays the foundations of mathematical analysis. It also introduces the theory of integer partitions, in this chapter.

There are several proofs of Euler's theorem. The one we proceed to sketch uses generating functions. It relies on observing that $\sum_{n=0}^{\infty} q(n)x^n$ can be represented as $\prod_{n=1}^{\infty}(1 + x^n)$: At least formally, this product can be expanded by picking from each factor $1 + x^n$ one of the two summands, with the understanding that in each product, all but finitely many times the summand 1 is the chosen one. Grouping together like terms, the coefficient of x^n in this expansion counts the number of ways the exponent n can be formed as a sum of distinct terms, which is precisely $q(n)$. For example, note that the only products that result in an x^5 term are $1 \cdot 1 \cdot 1 \cdot 1 \cdot x^5 = x \cdot 1 \cdot 1 \cdot x^4 = 1 \cdot x^2 \cdot x^3$, where in each product we have omitted the infinitely many remaining 1s.

Now, note that $\prod_{n=1}^{\infty}(1 + x^n) = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n} = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}$, the latter equality holding because all numerators cancel out and the only denominators that survive are the ones with odd degree. Expanding this product reveals that it is the generating function for partitions into odd parts:

$$\begin{aligned}
\prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} &= \prod_{n=1}^{\infty}(1 + x^{2n-1} + x^{2(2n-1)} + x^{3(2n-1)} + \cdots) \\
&= (1 + x + x^{2\cdot 1} + \cdots)(1 + x^3 + x^{2\cdot 3} + \cdots)(1 + x^5 + x^{2\cdot 5} + \cdots)\cdots \\
&= (1 + x + x^1x^1 + \cdots)(1 + x^3 + x^3x^3 + \cdots)(1 + x^5 + x^5x^5 + \cdots)\cdots \\
&= 1 + x + x^1x^1 + (x^1x^1x^1 + x^3) + (x^1x^1x^1x^1 + x^1x^3) + \cdots.
\end{aligned}$$

The argument above can be readily formalized either in terms of formal power series expansions or in terms of "genuine" power series (upon arguing that the series involved converge for $|x| < 1$).

Many other interesting results are known for q and other partition functions, see [1] for an introduction. These results are established by a wide variety of techniques, including combinatorial counting arguments, Ferrers diagrams, and others, and some are quite sophisticated, involving detailed analytical arguments, which entered the picture thanks to the joint work of Hardy and Ramanujan at the beginning of the twentieth century.

A small detour

The first-named author cannot help but mention that some of his own work involves the study of partitions, in this case partitions of infinite sets. This is part of the area of set theory called the partition calculus. As a simple example of the sort of problems one considers here, readers familiar with the distinction between countable and uncountable sets may enjoy verifying the following: Suppose the set $\mathbb{R}$ of reals is partitioned into countably many pieces, $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$. Then at least one of the sets A_n contains an infinite increasing sequence. Note the result fails for $\mathbb{Q}$ in place of $\mathbb{R}$ (we can split $\mathbb{Q}$ into countably many singletons). On the other hand, the result is not simply an artifact of $\mathbb{R}$ being uncountable (which ensures that one of the A_n is also uncountable), since not every uncountable ordered set contains an infinite increasing sequence.

About Partiti's creator

We hope readers enjoy Partiti and the many other puzzles we anticipate seeing from Thinh. They can learn more about Thinh himself in a recent piece on his work by Will Shortz that ran in The New York Times [4]. Thinh's puzzle Bar Code appeared for 14 weeks in the Sunday Magazine section of The Times.

Thinh has shown us a large variety of different puzzles of his own creation, all of a somewhat mathematical flavor. We asked him a few questions in preparation for this note. He shared with us that he solves all his puzzles manually, and uses the time it takes him to estimate their difficulty. He has surprised himself a few times with how hard some of his creations turned out to be. Although the name "Partiti" is probably self-explanatory, most of his puzzle names are inspired by Japanese puzzles, of which he confesses to be a big fan.

Thinh advises readers interested in creating their own mathematical puzzles that it is necessary to build a basic foundation and to read about numbers and logic puzzles. It took him five years to acquire this foundation himself. He indicates that some of the examples he has submitted to this MAGAZINE are harder than the ones he has had featured in The New York Times, and hopes to publish a book of his own puzzles.

Acknowledgment We thank Thinh Lai for his enthusiasm and help with the preparation of this note.

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Summary. We introduce PARTITI, the puzzle that will run in this MAGAZINE this year, and use the opportunity to recall some basic properties of integer partitions.

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PARTITI PUZZLE

17	3	10	19	3	22
8	17	9	4	2	18
9	2	16	1	12	13
17	10	4	8	18	2
15	3	5	2	5	12
23	4	16	18	20	8

How to play. In each cell, place one or more distinct integers from 1 to 9 so that they sum to the value in the top left corner. No integer can be used more than once in horizontally, vertically, or diagonally adjacent cells.

The solution is on page 15.

—contributed by Lai Van Duc Thinh,
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