MR3656308 03E02 03E10 54A25 54H05
Caicedo, Andrés Eduardo (1-MR); **Hilton, Jacob** (4-LEED-SM)
★**Topological Ramsey numbers and countable ordinals. (English summary)**
Foundations of mathematics, 87–120, *Contemp. Math.*, 690, *Amer. Math. Soc.*, *Providence, RI*, 2017.

If $1 \le n < \omega$, κ is a cardinal, and β, α_i $(i < \kappa)$ are ordinals, then $\beta \to_{top} (\alpha_i)^n_{i<\kappa}$ denotes the statement that if $c\colon [\beta]^n \to \kappa$ is a coloring, then for some i there is a homeomorphism f of α_i onto some i-homogeneous subset of β. $\beta \to_{cl} (\alpha_i)^n_{i<\kappa}$ holds if f can be required to be an order isomorphism, as well. $P^{top}(\alpha_i)_{i<\kappa}$, $P^{cl}(\alpha_i)_{i<\kappa}$ (respectively, topological pigeonhole number and closed pigeonhole number) denote the minimal ordinals β with positive relation for $n = 1$, and $R^{top}(\alpha_i)_{i<\kappa}$, $R^{cl}(\alpha_i)_{i<\kappa}$ (respectively, topological Ramsey number and closed Ramsey number) are defined similarly for $n = 2$.

The authors give bounds for these in several cases. An algorithm is given for calculating $P^{cl}(\alpha_i)_{i<\kappa}$ complementing similar work for P^{top} by J. Hilton in [J. Symb. Log. **81** (2016), no. 2, 662–686; MR3519451].

The main theorem, a topological version of a theorem by P. Erdős and E. C. Milner [see Canad. Math. Bull. **15** (1972), 501–505; MR0332492; corrigendum, Canad. Math. Bull. **17** (1974), 305; MR0360284], is the following. If α, β are positive, countable ordinals, $1 < k < \omega$, and $\omega^{\omega^\alpha} \to_{top} (\omega^\beta, k)$ holds, then one has $\omega^{\omega^\alpha \cdot \beta} \to_{top} (\omega^\beta, k+1)^2$. This gives $\omega^{\omega^{\alpha k}} \to_{cl} (\omega^{\omega^\alpha}, k+1)^2$ $(\alpha < \omega_1,\ k < \omega)$ and so $R^{cl}(\alpha, k) < \omega_1$ for $\alpha < \omega_1$, $k < \omega$. Also, an inductive upper bound is given for R^{top} and R^{cl} using smaller values and P^{top}, P^{cl} (similarly to a classical inductive formula for the Ramsey numbers $R(n, m)$).

A large number of individual values are calculated or estimated. It is shown, for example, that $R^{top}(\omega+1, k+1) = R^{cl}(\omega+1, k+1) = \omega^k + 1$ $(1 \le k < \omega)$. Further, $\omega^\omega \to_{cl} (\omega^2, k)^2$ and $\omega^{\omega k} + 1 \to_{cl} (\omega^2+1, k+2)^2$ $(1 < k < \omega)$. Also, $\omega^2 \cdot 3 \le R^{top}(\omega \cdot 2, 3) \le \omega^3 \cdot 100$.

The proofs use a plethora of earlier results and proof techniques for the discrete case plus careful analyses of the arithmetic of ordinals represented in the Cantor normal form. Nevertheless, a full characterization of $\beta \to_{cl} (\alpha, k)^2$ for β, $\alpha < \omega_1$, $k < \omega$ seems far away.

{For the collection containing this paper see MR3656304}

Péter Komjáth

References

1. J. E. Baumgartner, *Partition relations for countable topological spaces*, J. Combin. Theory Ser. A **43** (1986), no. 2, 178–195, DOI 10.1016/0097-3165(86)90059-2. MR867644 MR0867644
2. A. E. Caicedo. Teoría de Ramsey de ordinales contables muy pequeños. Submitted. Available at https://andrescaicedo.wordpress.com/papers/, 2014.
3. A. E. Caicedo. Partition calculus of small countable ordinals. A survey of results of Haddad and Sabbagh. Preprint, 2015.
4. P. Erdős and E. C. Milner, *A theorem in the partition calculus*, Canad. Math. Bull. **15** (1972), 501–505, DOI 10.4153/CMB-1972-088-1. MR0332492 MR0332492
5. P. Erdős and R. Rado, *A problem on ordered sets*, J. London Math. Soc. **28** (1953), 426–438, DOI 10.1112/jlms/s1-28.4.426. MR0058687 MR0058687

6. P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489, DOI 10.1090/S0002-9904-1956-10036-0. MR0081864 MR0081864
7. P. Erdős and R. Rado, *Partition relations and transitivity domains of binary relations*, J. London Math. Soc. **42** (1967), 624–633, DOI 10.1112/jlms/s1-42.1.624. MR0218248 MR0218248
8. J. Flum and J. C. Martínez, *On topological spaces equivalent to ordinals*, J. Symbolic Logic **53** (1988), no. 3, 785–795, DOI 10.2307/2274571. MR960998 MR0960998
9. H. Friedman, *On closed sets of ordinals*, Proc. Amer. Math. Soc. **43** (1974), 190–192. MR0327521 MR0327521
10. A. Hajnal, *Some results and problems on set theory* (English, with Russian summary), Acta Math. Acad. Sci. Hungar. **11** (1960), 277–298. MR0150044 MR0150044
11. A. Hajnal, *Remarks on the theorem of W. P. Hanf*, Fund. Math. **54** (1964), 109–113. MR0160734 MR0160734
12. J. Hilton, *The topological pigeonhole principle for ordinals*, J. Symb. Log. **81** (2016), no. 2, 662–686, DOI 10.1017/jsl.2015.45. MR3519451 MR3519451
13. A. Hajnal, I. Juhász, and W. Weiss, *Partitioning the pairs and triples of topological spaces*, Topology Appl. **35** (1990), no. 2-3, 177–184, DOI 10.1016/0166-8641(90)90103-9. MR1058798 MR1058798
14. A. Hajnal and J. A. Larson, *Partition relations*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 129–213, DOI 10.1007/978-1-4020-5764-9 3. MR2768681 MR2768681
15. L. Haddad and G. Sabbagh, *Sur une extension des nombres de Ramsey aux ordinaux* (French), C. R. Acad. Sci. Paris Sér. A-B **268** (1969), A1165–A1167. MR0246778 MR0246778
16. L. Haddad and G. Sabbagh, *Calcul de certains nombres de Ramsey généralisés* (French), C. R. Acad. Sci. Paris Sér. A-B **268** (1969), A1233–A1234. MR0248030 MR0248030
17. L. Haddad and G. Sabbagh, *Nouveaux résultats sur les nombres de Ramsey généralisés* (French), C. R. Acad. Sci. Paris Sér. A-B **268** (1969), A1516–A1518. MR0248031 MR0248031
18. A. L. Jones, *A short proof of a partition relation for triples*, Electron. J. Combin. **7** (2000), Research Paper 24, 9. MR1755613 MR1755613
19. Albin L. Jones. Even more on partitioning triples of countable ordinals. Proc. Amer. Math. Soc., to appear. MR2262926
20. V. Kieftenbeld and B. Löwe. A classification of ordinal topologies. Unpublished. Available at https://www.illc.uva.nl/Research/Publications/PP-2006-57.text.pdf, 2006.
21. A. H. Kruse, *A note on the partition calculus of P. Erdős and R. Rado*, J. London Math. Soc. **40** (1965), 137–148. MR0214476 MR0214476
22. R. Laver, *Partition relations for uncountable cardinals* $\leq 2^{\aleph_0}$, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, North-Holland, Amsterdam, 1975, pp. 1029–1042. Colloq. Math. Soc. Janós Bolyai, Vol. 10. MR0371652 MR0371652
23. J. A. Larson and W. J. Mitchell, *On a problem of Erdős and Rado*, Ann. Comb. **1** (1997), no. 3, 245–252, DOI 10.1007/BF02558478. MR1630775 MR1630775
24. O. Mermelstein. Calculating the closed ordinal Ramsey number $R^{cl}(\omega \cdot 2, 3)^2$. Preprint. Available at https://arxiv.org/abs/1702.03878, 2017.
25. E. C. Milner, *A finite algorithm for the partition calculus*, Proceedings of the Twenty-Fifth Summer Meeting of the Canadian Mathematical Congress (Lakehead Univ., Thunder Bay, Ont., 1971), Lakehead Univ., Thunder Bay, Ont., 1971, pp. 117–128. MR0332507 MR0332507

26. C. Piña, *A topological Ramsey classification of countable ordinals*, Acta Math. Hungar. **147** (2015), no. 2, 477–509, DOI 10.1007/s10474-014-0413-5. MR3420590 MR3420590
27. S. Rosenberg. Infinite closed monochromatic subsets of a metric space. Preprint. Available at https://arxiv.org/abs/1508.02366, 2015.
28. R. Schipperus, *Countable partition ordinals*, Ann. Pure Appl. Logic **161** (2010), no. 10, 1195–1215, DOI 10.1016/j.apal.2009.12.007. MR2652192 MR2652192
29. R. Schipperus, *The topological Baumgartner-Hajnal theorem*, Trans. Amer. Math. Soc. **364** (2012), no. 8, 3903–3914, DOI 10.1090/S0002-9947-2012-04990-7. MR2912439 MR2912439
30. W. Sierpiński, *Sur un problème de la théorie des relations* (French), Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **2** (1933), no. 3, 285–287. MR1556708 MR1556708
31. E. Specker, *Teilmengen von Mengen mit Relationen* (German), Comment. Math. Helv. **31** (1957), 302–314. MR0088454 MR0088454
32. S. Todorčević, *Forcing positive partition relations*, Trans. Amer. Math. Soc. **280** (1983), no. 2, 703–720, DOI 10.2307/1999642. MR716846 MR0716846
33. S. Todorčević. A partition property of spaces with point-countable bases. Unpublished. 1 pg., June 1996.
34. W. Weiss, *Partitioning topological spaces*, Mathematics of Ramsey theory, Algorithms Combin., vol. 5, Springer, Berlin, 1990, pp. 154–171, DOI 10.1007/978-3-642-72905-8 11. MR1083599 MR1083599
35. T. V. Weinert, *Idiosynchromatic poetry*, Combinatorica **34** (2014), no. 6, 707–742, DOI 10.1007/s00493-011-2980-1. MR3296182 MR3296182
36. N. H. Williams, *Combinatorial set theory*, Studies in Logic and the Foundations of Mathematics, vol. 91, North-Holland Publishing Co., Amsterdam, 1977. MR3075383 MR3075383

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

Published on October 09, 2017