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Downward transference of mice and universality of local core models. (English summary)
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The goal of this paper is to study inner models $\mathbf{M}$ of the set-theoretic universe $\mathbf{V}$, assuming that there is an agreement of cardinals between $\mathbf{M}$ and $\mathbf{V}$, for instance that $\omega_2^{\mathbf{M}} = \omega_2$.

The results are motivated by problems related to forcing axioms. B. Veličković showed that if Martin's Maximum MM holds and $\mathbf{M}$ is a (proper class) inner model that computes ω_2 correctly, then $\mathbb{R} \subseteq \mathbf{M}$ [Adv. Math. **94** (1992), no. 2, 256–284; MR1174395]. Moreover, A. E. Caicedo and Veličković obtained the same conclusion if, rather than MM in $\mathbf{V}$, the bounded proper forcing axiom BPFA is assumed to hold in both $\mathbf{M}$ and $\mathbf{V}$ [Math. Res. Lett. **13** (2006), no. 2-3, 393–408; MR2231126], while S.-D. Friedman showed that PFA in $\mathbf{V}$ is not sufficient [Ann. Japan Assoc. Philos. Sci. **19** (2011), 29–36; MR2857735].

The authors suggest that there might be ZFC results behind these phenomena. A possible way of formalizing this is to show that the large cardinal strength coded by reals in $\mathbf{V}$ is also present in $\mathbf{M}$. In fact, they make the following general conjecture.

Conjecture 1.1. Let r be a 1-small sound (iterable) mouse that projects to ω. Assume that $\mathbf{M}$ is an inner model and that $\omega_2^{\mathbf{M}} = \omega_2$. Then $r \in \mathbf{M}$.

The main result is a major step towards proving this conjecture. To state this result, we assume that $\mathbf{M}$ is an inner model and define the following notation. Let

$$\mathcal{P}_\delta(\lambda) = \{x \subseteq \lambda \mid \operatorname{otp}(x) < \delta \wedge x \cap \delta \text{ is transitive}\}$$

and

$$S_\delta = \{x \in \mathcal{P}_\delta(\delta^+) \cap \mathbf{M} \mid \operatorname{cf}^{\mathbf{M}}(x \cap \delta) > \omega\}$$

for any regular cardinal δ and any cardinal $\lambda \geq \delta$.

Theorem 1.6. Assume that $\mathbf{M}$ is a proper class inner model, and that δ is a regular cardinal in the sense of $\mathbf{V}$.

(a) Granting that, in $\mathbf{V}$, there is no proper class inner model with a Woodin cardinal, $\delta > \omega_1$, and S_δ is stationary, then the initial segment $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is universal for all iterable premice in $\mathbf{V}$ of cardinality less than δ.

(b) Granting that, in $\mathbf{M}$, $0^\P$ does not exist, and $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary, then: If $\delta > \omega_1$, then the initial segment of $\mathbf{K}^{\mathbf{M}} \parallel \delta$ is universal for all iterable premice in $\mathbf{V}$ of cardinality less than δ. Moreover, if $\delta = \omega_1$, then $\mathbf{K}^{\mathbf{M}} \parallel \omega_2$ is universal for all countable iterable premice in $\mathbf{V}$.

Here $0^\P$ denotes the sharp for an inner model with a strong cardinal. Theorem 1.6 implies for instance that for any proper class inner model $\mathbf{M}$ of ZFC with $\omega_2^{\mathbf{M}} = \omega_2$, every mouse projecting to ω and not past $0^\P$ belongs to $\mathbf{M}$.

The proofs of the main result are based on R. B. Jensen's proof of the universality of $\mathbf{K}^c$ described in ["Addendum to 'A new fine structure for higher core models'", unpublished manuscript, 1998] and use difficult inner model-theoretic techniques. Finally, the authors discuss several interesting ideas on how the current proofs could be extended to prove larger parts of Conjecture 1.1 and what the obstacles are to reaching this goal.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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