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**Topological Ramsey numbers and countable ordinals.** (English) Zbl 06767144

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Summary: We study the topological version of the partition calculus in the setting of countable ordinals. Let $\alpha$ and $\beta$ be ordinals and let $k$ be a positive integer. We write $\beta \to_{top} (\alpha, k)^2$ to mean that, for every red-blue coloring of the collection of 2-sized subsets of $\beta$, there is either a red-homogeneous set homeomorphic to $\alpha$ or a blue-homogeneous set of size $k$. The least such $\beta$ is the topological Ramsey number $R^{top}(\alpha, k)$.

We prove a topological version of the Erdös-Milner theorem, namely that $R^{top}(\alpha, k)$ is countable whenever $\alpha$ is countable. More precisely, we prove that $R^{top}(\omega^{\omega^\beta}, k+1) \leq \omega^{\omega^{\beta \cdot k}}$ for all countable ordinals $\beta$ and finite $k$. Our proof is modeled on a new easy proof of a weak version of the Erdős-Milner theorem that may be of independent interest.

We also provide more careful upper bounds for certain small values of $\alpha$, proving among other results that $R^{top}(\omega+1, k+1) = \omega^k+1$, $R^{top}(\alpha, k) < \omega^\omega$ whenever $\alpha < \omega^2$, $R^{top}(\omega^2, k) \leq \omega^\omega$ and $R^{top}(\omega^2+1, k+2) \leq \omega^{\omega \cdot k} + 1$ for all finite $k$.

Our computations use a variety of techniques, including a topological pigeonhole principle for ordinals, considerations of a tree ordering based on the Cantor normal form of ordinals, and some ultrafilter arguments.

For the entire collection see [Zbl 1367.03010].

### MSC:

03E02 Partition relations
03E10 Ordinal and cardinal arithmetic
54A25 Cardinality properties of topological spaces

### Keywords:

partition calculus; countable ordinals

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### References:

[1] Baumgartner, James E., Partition relations for countable topological spaces, J. Combin. Theory Ser. A, 0097-3165, 43, 2, 178-195, (1986) · Zbl 0647.05005 · doi:10.1016/0097-3165(86)90059-2

[2] ] A. E. Caicedo. Teoria de Ramsey de ordinales contables muy pequenos. Submitted. Available at https://andrescaicedo.wordpress.com/pap 2014.

[3] ] A. E. Caicedo. Partition calculus of small countable ordinals. A survey of results of Haddad and Sabbagh. Preprint, 2015.

[4] Erd\H os, P.; Milner, E. C., A theorem in the partition calculus, Canad. Math. Bull., 0008-4395, 15, 501-505, (1972) · Zbl 0271.04003 · doi:10.4153/CMB-1972-088-1

[5] Erd\H os, P.; Rado, R., A problem on ordered sets, J. London Math. Soc., 0024-6107, 28, 426-438, (1953) · Zbl 0051.04003 · doi:10.1112/jlms/s1-28.4.426

[6] Erd\H os, P.; Rado, R., A partition calculus in set theory, Bull. Amer. Math. Soc., 0002-9904, 62, 427-489, (1956) · Zbl 0071.05105 · doi:10.1090/S0002-9904-1956-10036-0

[7] Erd\H os, P.; Rado, R., Partition relations and transitivity domains of binary relations, J. London Math. Soc., 0024-6107, 42, 624-633, (1967) · Zbl 0204.00905 · doi:10.1112/jlms/s1-42.1.624

[8] Flum, J{\"o}rg; Mart{\'{\i }}nez, Juan Carlos, On topological spaces equivalent to ordinals, J. Symbolic Logic,

0022-4812, 53, 3, 785-795, (1988) · Zbl 0655.03026 · doi:10.2307/2274571

[9] Friedman, Harvey, On closed sets of ordinals, Proc. Amer. Math. Soc., 0002-9939, 43, 190-192, (1974) · Zbl 0299.04003

[10] Hajnal, A., Some results and problems on set theory, Acta Math. Acad. Sci. Hungar., 0001-5954, 11, 277-298, (1960) · Zbl 0106.00901

[11] Hajnal, A., Remarks on the theorem of W. P. Hanf, Fund. Math., 0016-2736, 54, 109-113, (1964) · Zbl 0119.01505

[12] Hilton, Jacob, The topological pigeonhole principle for ordinals, J. Symb. Log., 0022-4812, 81, 2, 662-686, (2016) · Zbl 1370.03062 · doi:10.1017/jsl.2015.45

[13] Hajnal, A.; Juh{\'a}sz, I.; Weiss, W., Partitioning the pairs and triples of topological spaces, Topology Appl., 0166-8641, 35, 2-3, 177-184, (1990) · Zbl 0703.04001 · doi:10.1016/0166-8641(90)90103-9

[14] Hajnal, Andr{\'a}s; Larson, Jean A., Partition relations.Handbook of set theory. Vols. 1, 2, 3, 129-213, (2010), Springer, Dordrecht · Zbl 1198.03048 · doi:10.1007/978-1-4020-5764-9_3

[15] Haddad, Labib; Sabbagh, Gabriel, Sur une extension des nombres de Ramsey aux ordinaux, C. R. Acad. Sci. Paris S\'er. A-B, 268, A1165-A1167, (1969) · Zbl 0181.30302

[16] Haddad, Labib; Sabbagh, Gabriel, Calcul de certains nombres de Ramsey g\'en\'eralis\'es, C. R. Acad. Sci. Paris S\'er. A-B, 268, A1233-A1234, (1969) · Zbl 0181.30303

[17] Haddad, Labib; Sabbagh, Gabriel, Nouveaux r\'esultats sur les nombres de Ramsey g\'en\'eralis\'es, C. R. Acad. Sci. Paris S\'er. A-B, 268, A1516-A1518, (1969) · Zbl 0181.30304

[18] Jones, Albin L., A short proof of a partition relation for triples, Electron. J. Combin., 1077-8926, 7, Research Paper 24, 9 pp., (2000) · Zbl 0945.03067

[19] ] Albin L. Jones. Even more on partitioning triples of countable ordinals. Proc. Amer. Math. Soc., to appear. · Zbl 1110.03033

[20] ] V.Kieftenbeld and B.Lowe. A classification of ordinal topologies. Unpublished. Available at https://www.illc.uva.nl/Research/Publications 2006-57.text.pdf, 2006.

[21] Kruse, Arthur H., A note on the partition calculus of P. Erd\H os and R. Rado, J. London Math. Soc., 0024-6107, 40, 137-148, (1965) · Zbl 0163.25002

[22] Laver, R., Partition relations for uncountable cardinals $\leq 2^{\aleph _{0}}$.Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erd\H os on his 60th birthday), Vol. II, 1029-1042. Colloq. Math. Soc. Jan\'os Bolyai, Vol. 10, (1975), North-Holland, Amsterdam

[23] Larson, Jean A.; Mitchell, William J., On a problem of Erd\H os and Rado, Ann. Comb., 0218-0006, 1, 3, 245-252, (1997) · Zbl 0895.05043 · doi:10.1007/BF02558478

[24] ] O. Mermelstein. Calculating the closed ordinal Ramsey number $R^{cl}(2,3)^{2}$. Preprint. Available at https://arxiv.org/abs/1702.0387 2017.

[25] Milner, E. C., A finite algorithm for the partition calculus.Proceedings of the Twenty-Fifth Summer Meeting of the Canadian Mathematical Congress, Lakehead Univ., Thunder Bay, Ont., 1971, 117-128, (1971), Lakehead Univ., Thunder Bay, Ont.

[26] Pi{\~n}a, C., A topological Ramsey classification of countable ordinals, Acta Math. Hungar., 0236-5294, 147, 2, 477-509, (2015) · Zbl 1374.03030 · doi:10.1007/s10474-014-0413-5

[27] ] S.Rosenberg. Infinite closed monochromatic subsets of a metric space. Preprint. Available at https://arxiv.org/abs/1508.02366, 2015.

[28] Schipperus, Rene, Countable partition ordinals, Ann. Pure Appl. Logic, 0168-0072, 161, 10, 1195-1215, (2010) · Zbl 1225.03058 · doi:10.1016/j.apal.2009.12.007

[29] Schipperus, Ren{\'e}, The topological Baumgartner-Hajnal theorem, Trans. Amer. Math. Soc., 0002-9947, 364, 8, 3903-3914, (2012) · Zbl 1291.03080 · doi:10.1090/S0002-9947-2012-04990-7

[30] Sierpi{\'n}ski, Waclaw, Sur un probl\`eme de la th\'eorie des relations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2), 0391-173X, 2, 3, 285-287, (1933) · Zbl 0007.09702

[31] Specker, Ernst, Teilmengen von Mengen mit Relationen, Comment. Math. Helv., 0010-2571, 31, 302-314, (1957) · Zbl 0080.03703

[32] Todor{\v {c}}evi{\'c}, Stevo, Forcing positive partition relations, Trans. Amer. Math. Soc., 0002-9947, 280, 2, 703-720, (1983) · Zbl 0532.03023 · doi:10.2307/1999642

[33] ] S.Todorcevic. A partition property of spaces with point-countable bases. Unpublished. 1 pg., June 1996.

[34] Weiss, William, Partitioning topological spaces.Mathematics of Ramsey theory, Algorithms Combin. 5, 154-171, (1990), Springer, Berlin · Zbl 0755.54002 · doi:10.1007/978-3-642-72905-8_11

[35] Weinert, Thilo V., Idiosynchromatic poetry, Combinatorica, 0209-9683, 34, 6, 707-742, (2014) · Zbl 1340.03013 · doi:10.1007/s00493-011-2980-1

[36] Williams, Neil H., Combinatorial set theory, 0-7204-0722-2, Studies in Logic and the Foundations of Mathematics 91, xi+208 pp., (1977), North-Holland Publishing Co., Amsterdam · Zbl 0362.04008