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Inner-model reflection principles. (English summary)
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In the spirit of the current trend in set theory of considering structural propositions about inner, outer, and forcing models, this paper discusses the *inner-model reflection principle*: If a formula $\varphi(a)$ holds in V then it holds in a proper inner model W with $a \in W$. Related is the stronger *ground-model reflection principle* in which one requires the W to be a transitive inner model of which V is a set-forcing extension. I refer to the first principle as IMR and the second as GMR. The first author raised the issue of having such principles that express "width reflection", which is in contrast to "height reflection" as given by the usual Reflection Principle derivable in ZFC. This paper consists of straightforward observations or applications of procedures and results drawing from the various authors' expertise.

With respect to forcing, it is observed that every model of ZFC has a class-forcing extension satisfying GMR, with the argument showing that IMR and GMR are each conservative over ZFC for Π_1 set assertions. With respect to large cardinals, it is observed, e.g., that having a proper class of measurable cardinals implies IMR, and that by forcing that preserves measurable cardinals, GMR can be made to fail. A sharper observation made is that "Ord is Ramsey" implies IMR, and that the hypothesis here can be further weakened. In the most intricate argument of the paper, it is shown that the consistency strength of having a fine-structural extender model satisfying GMR is exactly that of having a proper class of Woodin cardinals. Next given are observations having to do with other known forcing and inner-model principles and forcing axioms. Finally, having noted that GMR is expressible as a first-order schema, arguments are given toward the assertion that IMR cannot be so expressible.

This paper illustrates how modern set theory has developed ready techniques and ways of thinking that can be brought to bear to illuminate structural assertions about the universe of sets. *A. Kanamori*

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